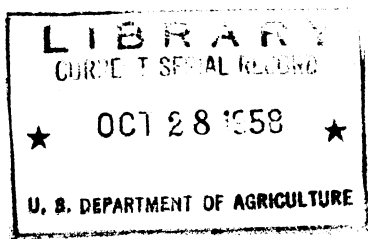


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# CARRYOVER LEVELS for Grains



**A method for determining  
amounts that are optimal  
under specified conditions**



## FOREWORD

The Department of Agriculture has often been urged to do more "basic" or "fundamental" research. It has also been urged to do more research to provide information needed in formulating policy and operating programs. The study here reported responds to both these recommendations. It is an analysis of fundamental economic principles for managing a public storage program for grains.

Since it is basic research, it does not tell how many bushels of wheat or corn should be placed in storage or withdrawn from storage this year or next year. Rather, it develops rules that the administrator of a grain storage program—with the help of his economic and statistical advisors—can apply for deciding how much to store or release in any given year.

A public storage program may have a single objective or a combination of objectives. Among the purposes that have been talked about for such programs are stabilization of supplies from year to year, stabilization of prices, stabilization of producers' returns, safeguarding against national emergency, getting the greatest return for producers, supplying consumers' needs at lowest cost, and maximizing the "public benefit." The example worked out in this study assumes the last-named objective. Also, it assumes that the area under the demand curve can be used as an estimate of public benefit. In this particular case, the study shows that the optimum storage program would be identical with the program we would expect private industry to carry out if there were pure and perfect competition.

But while the example is worked out in terms of the area under the demand curve, the general principles of this study could be applied in any case where both the objectives and the cost of the program can be stated in definite and quantitative terms. Given any sort of "value function," this study shows how to take into account such factors as the initial supply on hand, prospective acreage and variability in yield in future years, characteristics of demand for the product, and costs of storage, including both handling costs and interest on the money invested.


This study was carried on at the University of Chicago in large part under contract with the United States Department of Agriculture. It makes a substantial addition to our understanding of relationships relevant to sound management of a storage program.

FREDERICK V. WAUGH,  
*Director, Agricultural Economics Division.*

WASHINGTON, D. C.

Issued October 1958

# **CARRYOVER LEVELS for Grains**



**A method for determining  
amounts that are optimal  
under specified conditions**

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# CARRYOVER LEVELS FOR GRAINS

## A Method for Determining Amounts that are Optimal Under Specified Conditions

By ROBERT L. GUSTAFSON, *Assistant Professor of Economics, University of Chicago*<sup>1</sup>

### SUMMARY

The idea that annual fluctuations in supplies of grains and other storable agricultural commodities can or should be evened out through the medium of year to year storage is thousands of years old. Despite the existence of a considerable body of literature on the subject, however, the following important questions have not been fully and rigorously answered: (1) In any year—or better, in each year of a contemplated period of years—exactly how much grain should be put into or removed from storage, given the best available information on the conditions that are relevant to making such a decision? (2) Given the quantity of grain that is to be stored in the nation as a whole in any year, what is the best regional distribution of that quantity of storage, that is, where should the grain be kept and in what amounts?

For complete mathematical rigor, both the national and the regional aspects of the storage question should be answered simultaneously. A mathematical solution for optimal multiregional rules is given. It turns out, however, that even for the simplest case—that is, a 2-year, 2-region model—the computations would be formidable except on a high-speed electronic computer. The bulletin, therefore, is concerned chiefly with methods of determining optimal storage policy at the national level.

Decisions made by farmers and the trade with respect to quantities to be carried over from one period to another chiefly depend on their expectations of relative current and future prices. Decisions on the part of governmental agencies with respect to storage policies generally reflect other sorts of considerations. Here the goal may be to even out supplies, to assure minimum stocks to meet emergency requirements, or to maintain stable returns to producers. The examples given in this bulletin relate chiefly to obtaining a storage policy that will result in the maximum net benefit to the general public, when total benefit is measured as the area under the demand curve, although the general approach used could be applied to several alternative goals.

This bulletin is concerned basically with procedures that can be used to even out supplies of grain by varying the quantity carried over from year to year. In actual practice, stabilization proposals

<sup>1</sup> This work was started and in considerable part completed while the author was a Research Assistant at the University of Chicago. Richard J. Foote of the Agricultural Marketing Service gave substantial assistance in preparing the report. Helpful advice was received from several people at the University and in the Department of Agriculture, including in particular K. A. Fox, I. Herstein, D. G. Johnson, J. Marschak, T. W. Schultz, G. Tolley, and W. A. Wallis.

seldom rely only on storage. Instead, in times of surplus, use is also made of export subsidies or other export disposal programs and of domestic diversion programs. A research program currently in progress is considering the relative costs and gains to farmers and the general public of alternative combined programs, and of a storage policy designed primarily to support prices of particular commodities at desired levels as contrasted with the procedures developed in this bulletin, which are designed primarily to even out year-to-year supplies.

Before applying the methods, we must first make some judgment concerning the value to the general public of consuming alternative amounts of grain in various years. Here we are concerned essentially with the relative value obtained by consuming a fairly stable quantity of grain in each of several years, or of consuming the same total quantity over the entire period but in variable amounts from one year to the next. One way of making a rough estimate of the value of consuming a specific amount of grain is to take the area under a demand curve. This procedure frequently has been followed by economists in the past, and it is used in most of this report. However, the general approach used to derive the rules developed here can be applied to any method of measuring total value so long as this value can be expressed as some sort of function of the quantity consumed.

Having defined the total value to the general public in each year as a function of the quantity consumed in that year, we note that the quantity consumed in turn is equal to the initial supply, that is, production plus beginning stocks, minus the carryover. We next define *net* benefit in any year as the total value less costs of storage, including interest on the investment. In any given year, then, for a given level of initial supply, determination of the carryover determines each of the following: The quantity consumed (supply minus carryover), total value (a function of quantity consumed), cost of storage (a function of the carryover), and the net benefit (total value minus cost of storage). Thus, all of these variables, in particular the net benefit—with which we are primarily concerned—depend on or are determined by the level of initial supply and the level of carryover. Hence, if it is possible to specify some functional relationship between the carryover and the initial supply, then the relevant variables, including net benefit, are determined by the initial supply and the specified functional relationship. Such a relationship between supply and carryover we shall call a *storage rule*. It may be thought of as a table in which, for various possible different levels of supply, the corresponding carryover is given; or as a graph on which the same information is specified; or in some cases possibly as a mathematical formula.

The first question that suggests itself is whether it is possible to specify, and to determine the values of, such storage rules. One of the objects of this bulletin is to show that it is not only possible, but indeed necessary, to specify such relationships or rules, under the conditions and objectives stipulated; and also to show how the values of the rules can be obtained.

A *storage policy* for a period of years is defined as a set of storage rules, one for each year. If we consistently follow a set of storage rules, the net benefit in any year depends on the initial supply and the rule for that year. The supply is equal to beginning stocks plus

production. Beginning stocks, in turn, depend on the supply and the storage rule applied in the *preceding* year; hence it is necessary, in general, in analyzing the storage problem, to think in terms of sets of storage rules rather than an isolated rule or level of storage for a single year. Furthermore, levels of production in future years are in general not known; we get around this difficulty by making use of their estimated probability distributions. Using these probability distributions, it is conceptually possible, for a given set of storage rules, to obtain an average, or "mathematically expected" value for the net benefit in each future year. Applying an appropriate discount factor for each year to obtain the "present value" of the benefits, we add together and obtain the sum of discounted expected net benefits in all future years. An optimal storage policy, as given in this bulletin, is defined as that set of storage rules which maximizes the sum of discounted expected net benefits in all future years (or, in some cases, for a specified number of future years) for any given initial supply of grain in the initial year. The resulting storage rules state how much grain should be carried over into the following period given the initial supply for the current year.

Material in the bulletin is concerned primarily with methods for obtaining such rules; institutional, administrative, or statutory arrangements required to bring about the storage of such quantities are considered as outside its scope. It is shown, however, that, under certain conditions, the operations of private firms in a competitive market will result in the storage of quantities called for by the optimal rules. It should be noted that the methods for obtaining the rules developed here in general do not, for reasons of mathematical and computational feasibility, follow directly the procedure which might be suggested by the preceding paragraph; the discussion there is partly conceptual, the purpose being to outline the nature of the criterion of optimality; one of the objects of the bulletin is to present methods which are mathematically and computationally feasible and which will result in storage rules that do satisfy the criterion.

Methods by which alternative conditions can be incorporated into the rules are given. For example, allowance could be made for anticipated future variability in domestic demand if this could be measured. Likewise, the rules can be modified to maximize expected gains to a particular sector of the economy, such as farmers, if this appears desirable. Or they may be designed to stabilize prices rather than quantities utilized, as in the empirical examples shown. The general approach outlined is general enough to be applied to many different conditions and criteria. Thus, for example, the method of solution can readily be modified or extended to allow for the effects of foreign trade on the relevant conditions. However, for the sake of simplicity and because of some uncertainty about the accuracy of available estimates of future demand and supply conditions in foreign countries for grains (such as wheat), for which such estimates would be important, the empirical applications presented in this bulletin are confined to storageable commodities (namely, feed grains) for which net foreign trade is small in relation to total domestic use.

Storage rules for feed grains under 12 sets of alternative conditions are shown both in table and chart form. The charts are designed to show the effect on the rules of alternative assumptions about specified

conditions; they show that similarities, other than level, are greater than differences, even for the wide variety of conditions for which rules are computed. An equilibrium level for each rule is given in the table. It can be thought of as an average level around which the yearly carryovers over a long period tend to fluctuate for a given storage rule. The carryover that would be reached following two bumper crops also is given. Equilibrium carryovers for the corn equivalent of all corn, oats, and barley vary among the different rules from an approximate minimum workingstock level of 200 million bushels to 578 million bushels; the corresponding carryovers following two bumper crops vary from 774 to 1,656 million bushels.

Some knowledge of mathematics and probability calculus is required to derive the mathematical solutions upon which the storage rules are based; but, computation of the rules for particular empirical applications requires only numerical iterative procedures. In some cases, the required computations become extensive, and a shortcut method for approximating a rule under specified conditions is given. The shortcut method requires the use of relatively few arithmetic operations. Examples are shown to illustrate that the shortcut method results in a rule that is nearly the same as that computed by the more exact iterative method.

The basic principles that underlie the rules and some general conclusions with respect to storage that can be drawn from them are discussed in detail in nonmathematical terms; these sections of the bulletin require only a limited knowledge of mathematical symbols and operations. Mathematical solutions for the storage rules and certain special relationships that pertain to the storage problem then are given for the use of research workers who may have an interest in them.

## INTRODUCTION

From a standpoint of national policy, storage is important chiefly because of fluctuations in supply and demand through periods that extend up to several years in length. If neither production nor quantities needed for consumption varied, a uniform amount would be produced and consumed in each year and only minimum working stocks would be carried over from one year to the next. We all know that for grains, in particular, production changes greatly from year to year, reflecting chiefly variations in yield due to weather. In some recent years, production also has been affected to a significant extent by Government regulation of acreage. Year-to-year fluctuations in demand in general are less violent. But at times, as during or immediately following a major war, material changes may take place and may affect consumption for several years. Other factors, such as changes in taste and technology, are of perhaps greater importance in bringing about long-run changes in supply and demand.

This bulletin describes analytical techniques that deal with the question: For the nation as a whole, in any year, how much grain should be put into, or removed from, storage, given the best available information on conditions which are relevant to making such a decision. Results of applying the method to obtain storage rules for total feed grains in the United States which are optimal under specified alternative assumptions are shown.

The term "storage rule," as used in this bulletin, is a statement or formula that indicates, in any given year, how much should be carried over into the following period under specified conditions. An optimal set of storage rules is a set of rules that achieves specified desired objectives, which, because of uncertainty about relevant future conditions, are usually stated in terms of "expected values" of specified variables over a period of years.

Conditions that are relevant in making decisions with respect to storage may be divided into three categories: Those that relate to (1) production of the grain (supply), (2) utilization of the grain (demand), and (3) costs of storage and the interest rate (or the rate at which future costs and returns are discounted to get their present value). An explicit solution of the storage problem also must specify a criterion of optimality, by which is meant the end or objective in view. Because of the diversity of possible ends, *any* solution to the grain storage problem obtained by economic analysis alone must be a "proposed" solution; the actual choice of a policy must depend on the choice of objective. But with a *given* criterion of optimality, the economic analyst can provide what appears to be a "best" solution to the storage problem and the method outlined here is sufficiently general to be applied to many different criteria.

## OPTIMAL STORAGE RULES AT THE NATIONAL LEVEL

### A CRITERION OF OPTIMALITY

The criterion adopted here is the maximization of expected gain (or equivalently, the minimization of expected loss) to the general public arising from grain-storage operations over a period of years, where the "gain" is defined as specified on page 17, and where "expected" means "the mathematical expectation of" or "the mean of the probability distribution of." This criterion is believed to be generally acceptable, and it presumably underlies, implicitly or explicitly, most discussions of grain storage and related problems. The criterion can be discussed from three viewpoints:

1. Use of expected values implies that probability calculus is relevant; that is, that quantities which are not known with certainty can be treated as random variables, subject to probability distributions which are known or can be estimated. In the grain storage problem, as treated here, the main emphasis (at least initially) is on the element of uncertainty introduced by fluctuations in future yields per acre. On the basis of existing historical and technological data on yields, the construction of reasonably good estimates of probability distributions of future yields appears to be possible. To the extent that future fluctuations in other relevant variables (for example, demand, or acreage planted) can be treated as random (that is, subject to a known probability distribution), such fluctuations can be introduced explicitly into the solution.

2. The gain to be maximized is intended to be the gain to the general public, rather than to some particular sector of the economy, such as farmers or grain dealers.<sup>2</sup> However, the method of solution can be readily modified to maximize expected gains for any particular sector, if desired.

<sup>2</sup> For a discussion of the theory of storage and an examination of possible alternative objectives, see Johnson (6, ch. 10)\* and the accompanying bibliography.

\*Throughout this bulletin, italicized numbers in parentheses refer to Literature Cited, p. 64.

3. The criterion used here is stated in terms of net gains or losses arising from changes in quantities stored or utilized, rather than in terms of price stabilization. It is clear, however, that a program that partially or fully stabilizes quantities utilized is equivalent to a program that partially or fully stabilizes prices, given no change in the general price level or the level of demand. The carryover rules determined in this bulletin can easily be converted into equivalent price-setting rules.

The discussion in this bulletin pertains to the determination of desirable quantities to be stored under given circumstances, with little attention devoted to the institutional, administrative, or statutory arrangements required to bring about the storage of such quantities. Once the optimum amount to be stored is determined, the actual storage of that amount could be effectuated by various means, for example, by (1) outright governmental purchase or sale of the grain and storage by a governmental agency; (2) a price-setting, government-loan program to control private holdings of the grain; or perhaps (3), under some circumstances, simply improvement in information and stability of expectations in a free market for grains. Relations between "optimal" storage rules and storage activity that would tend to occur in an "idealized" free market are considered on page 48.

## PROPOSALS WITH RESPECT TO GRAIN STORAGE THAT HAVE BEEN MADE PREVIOUSLY

*The level-of-storage approach.*—The usual approach to the grain storage problem is in terms of a "level of storage." The analyst attempts to determine how much grain would have to be available from storage to offset the effects of certain contingencies such as a low yield or series of yields, or a war. He then estimates the average time for which the stocks would have to be held and the costs of holding the stocks over this period, and weighs such costs against the estimated benefits. Since the cost of holding sufficient stocks to offset any conceivable contingency, or even an actual unusual occurrence such as the droughts of the mid-1930's, turns out to be prohibitive, some compromise with the "ideal" of a complete offset must be made by an arbitrary method, and a "level" is arrived at which is adequate partially to offset certain contingencies. This approach has been used, for example, by Shepherd (10) and the authors of a recent Congressional report (12).

The author of this bulletin believes that such an approach is necessarily an inadequate solution to the storage problem. The reasons for the inadequacy may be summarized under the following points:

1. From the standpoint of an administrator who has to make actual storage decisions, a policy stated in terms of levels is almost meaningless. Under such a policy he knows only that he must operate in such a way that in the long run the amount in storage will tend toward the stipulated level, but this provides little guidance in determining how much to add to or subtract from storage in any given year. Suppose, for example, that stocks at the beginning of the current crop year are 10 percent below the recommended level, and the harvest in the current year is also 10 percent below normal. Should stocks be increased to bring them toward the recommended level (and if so, by how much), or should they be depleted further in order to augment the short crop (and if so, by how much)? A "level of storage" policy is of little help in answering such a question. What is needed is a rule of storage which indicates, for any specified level of stocks at the beginning of the year (carry-in) and any harvest, what amount should be added to or taken from

stocks during the year or, equivalently, what the level of stocks should be at the end of the year, that is, the carry-out.

2. The economic analyst is faced with an analogous situation, but the argument may be carried somewhat further. In this situation, we are trying to analyze how to divide an existing supply of grain between current and future use in such a way as to maximize the expected benefits to be derived from the use of the grain, both present and future, less expected costs. The answer to this question is a rule of storage, applicable this year. But the answer depends, in general, on how the grain is used in those future years and, in particular, on how it is distributed *among* those future years. Thus, it depends on the storage rules that are in operation in those years. We can say, then, that a storage policy intended to minimize losses or maximize benefits must be in the form of a *set* of storage rules. And, as we shall see, a straightforward, logical, and computational application of the criterion of maximizing the sum of discounted expected gains arising from storage operations results in such a set of rules. A storage policy stated in terms of a desired level of storage, on the other hand, never can be shown to be optimal, that is, no objective way exists for showing that one level is better than another.

Three modifications or additions to the above argument should be mentioned:

1. Anyone who discusses the determination of proper levels of storage obviously has in mind that the stocks will be manipulated in accordance with some kind of not-formally-defined "rule," that is, presumably, stocks generally will tend to build up in years of good crops and be depleted in years of poor crops. But this rule must be formally defined and quantified in order to make storage operations optimal.

2. Once the storage rules are determined, in some cases we can define and calculate, from the rules, what might be termed an "equilibrium" storage level, that is, a level toward which stocks tend, on the average and in the long run, when the rules are applied. In this way, storage rules can be related to, or compared with, what may be an intuitively more understandable concept of storage levels.

3. Suppose the criterion of maximizing expected gain is in fact rejected, and instead, for military reasons or otherwise, it is desired to have on hand at the end of a certain period (say 5 years) a specified level of reserve stocks. The problem is to determine the best way to build stocks to that level. Again we need a set of storage rules, and the method given here can be directly applied to such a problem. But a better way exists—as shown on page 57 to adjust storage policy to provide for the existence of military or other contingencies than simply building stocks to a predetermined level at the end of a period of years.

*Storage rules based on a plausible functional form.*—Granted, then, that the problem we face is the determination of good storage rules, where a rule for a given year is defined as a function which states, for each possible quantity of available supply, or harvest and carry-in, what should be the carry-out, the next question that arises is how to solve *that* problem. The simplest approach might appear to be (1) to assume some plausible functional form for the rule, (2) to calculate expected costs and benefits under the rule, such expected values being functions of the coefficients or parameters in the rule, and (3) to find those values of the parameters that minimize net expected losses or maximize net expected gains.

Two general objections to this procedure are:

1. We have no way of knowing whether an assumed form is really a good one, even though it may appear plausible. It is clearly preferable to have a procedure that requires no assumption as to form; as we shall see, such a procedure is, in fact, mathematically available.

2. Except in the simplest cases, computations required to find expected costs and gains as functions of parameters in the rule *and* of the current level of supply over a period longer than a few years may become quite extensive.

The following forms of rules have been suggested as having considerable "plausibility appeal:"

1. Let the carry-out be a fixed (determinable) proportion of the total supply, or of the total supply minus the minimum possible harvest [see Rosenblatt (8)]. Serious objections to the application of Rosenblatt's results in the determination of actual storage policy are outlined in Appendix Note 2. The criticisms there may be taken as illustrative of the dangers of assuming in advance a particular parametric or functional form for the rule.

2. Divide stocks into two categories, one for offsetting relatively minor or "normal" fluctuations in yields and the other, a reserve to be used only in case of serious drought, that is, when yields fall below some critical level. The assumption implicitly underlying such a policy is presumably that the utility- or demand-function is discontinuous. Such an assumption, however, can be directly incorporated into the solution outlined, beginning on page 40, without the necessity of setting up two categories for stocks.

*Storage rules for which the amount added depends on deviations in size of crop from normal.*—Another possible form of storage rule which has been considered is to make the amount added to storage a function of the amount by which the current year's harvest deviates from normal. The simplest function of this kind is a constant proportion. The idea underlying such a rule is that we face a certain variability of output which we want to transform into a smaller variability of quantity utilized. Such a transformation could theoretically be made by the kind of rule suggested. The objections to such a rule are:

1. It is operationally and analytically unsound, in the sense that it assumes that the decision as to how much grain should be added to storage this year can be made rationally while completely ignoring the amount already in storage.

2. Since the first few years of operation of the rule may be years of poor crops, in which case the rule will call for removing grain from storage, such a rule could be put into operation only at a time when existing stocks already are large, whereas a rule, to be generally useful, ought to be operational under any initial condition of supply. Furthermore, determination of the necessary level of initial stocks to make the rule workable must be probabilistic, since the initial stocks necessary to be completely certain that the rule could be worked for an indefinitely long period would be indefinitely large. Moreover, no obvious criterion exists for determining what should be the level of probability which one is willing to stipulate for the workability of the rule. (For further details, and a concrete example, see Appendix Note 3.)

3. Under a rule of this kind, an error in the estimate of the probability distribution of yields or its equivalent, an undetected change in the conditions of production, can lead to a system that "runs away." For example, if the estimate of the mean of the distribution is too low, stocks tend to build up indefinitely, whereas if the estimate is too high, stocks tend to decline to zero.

*An approach based on an idealized free market.*—Another possible approach to the storage problem is to construct a model designed to approximate the working of an idealized free market for grains, that is, a market in which all stocks are held by private firms, operating under perfect competition and maximizing expected profits. In a later section we see that, under certain conditions, the aggregate amounts stored in such a market can be calculated, using directly the methods presented in that section. Under these conditions, the rule becomes a description of market behavior instead of a means for decision making. The results can be used either (1) as the basis of an optimal rule of storage, assuming that what happens under the conditions outlined is desirable for the general public, or (2) as a basis for estimating the extent to which aggregate amounts stored under

actual (historical) market conditions have deviated from the amounts that would have been stored under the so-called "ideal" conditions.

### THE STORAGE PROBLEM STATED MORE PRECISELY

"Storage" throughout this discussion means year-to-year carryover, the presumption being that distribution of the product among years is the serious problem, whereas distribution within a year, given the total amount to be utilized during the year, is relatively trivial from a policy viewpoint. At the beginning of a given crop year (say on October 1 for corn, or July 1 for wheat) we know the amount of carryover from the preceding year ( $C_{t-1}$ ), and we can estimate fairly accurately the amount of the crop in that year ( $X_t$ ).<sup>3</sup> The total supply ( $S_t$ ) is the quantity available for utilization and carryover. The problem is to determine what the carryover should be at the end of the given year ( $C_t$ ), given the relevant conditions of demand, supply, cost of storage, and the interest rate. The quantity utilized ( $Y_t$ ) is, of course, simultaneously determined, as is the amount added to or subtracted from storage ( $C_t - C_{t-1}$ ). These relationships are expressed by the equations:

$$S_t = C_{t-1} + X_t \quad (1)$$

$$Y_t = S_t - C_t \quad (2)$$

$$= C_{t-1} + X_t - C_t \quad (3)$$

A "rule of storage," as used here, is simply a function ( $\theta_t$ ) which explicitly states the way in which  $C_t$  depends on  $C_{t-1}$  and  $X_t$ , that is:

$$C_t = \theta_t(C_{t-1}, X_t) \quad (4)$$

At this point we do not specify anything about the nature of this functional relationship. Later we see that most, if not all, optimal storage rules are nonlinear and that the algebraic expression of the relationship is moderately complex. A "storage policy" for a period of  $n$  years ( $t=1, \dots, n$ , where the current year is designated as 1) may be defined as a set of storage rules for those years ( $\theta_1, \dots, \theta_n$ ). Our problem, then, is that of finding a "good" policy for a given number of years ( $n \geq 2$ ). Storage rules or policies which are optimal under stated conditions are designated by a circumflex, thus:  $\hat{\theta}_t$  or  $(\hat{\theta}_1, \dots, \hat{\theta}_n)$ .

We actually may be primarily interested only in what to do in the current year ( $\hat{\theta}_1$ ), but determination of the best  $\theta_1$  in general depends upon  $\theta_2, \dots, \theta_n$ , so they also must be determined. Under the assumption that all relevant conditions and criteria are unchanging through time, sometimes referred to as an assumption of "stationarity," we have  $\hat{\theta}_1 = \hat{\theta}_2 = \dots$  *ad infinitum*, and the problem is to determine the best single rule  $\hat{\theta}$ , to be applied each year.

<sup>3</sup> For a list of the important symbols used, see Appendix note 1. Each symbol is defined, however, as it is introduced.

## SOME SIMPLIFYING RESTRICTIONS

This study initially was primarily concerned with storage as a means of offsetting fluctuations in yield. To simplify the analysis, we initially assume that the following are known with certainty: (1) The basic demand curve for the grain, (2) the cost of storage for various quantities stored, and (3) the acreage to be planted. We need not specify that the conditions are the same in each year but only that, if they do change, we know how they will change. Ignoring random or unpredictable fluctuations in acreage initially can be justified in part by the fact that, prior to price support programs, the effect on production of changes in acreage for most grains was small relative to the effect of fluctuations in yields.<sup>4</sup> The effect on optimal storage rules of introducing random or unpredictable fluctuations in demand or acreage into the solution is discussed on page 51.

For purposes of facilitating both analysis and discussion, we first consider a desirable storage policy for the country as a whole, that is, we initially ignore the existence of interregional differences and relationships. To do this, we set up two forms of restrictions as a framework for our analysis. The first form, designated as restriction I, can be stated in two alternative ways; the second form, designated as restriction II, can be stated in three alternative ways. These alternative statements are not necessarily equivalent, but any one of them will satisfy the requirement in each case. Nor are these conditions necessary, but only sufficient; one easily could think of other statements of conditions which would satisfy the requirements.

*Restriction I.*—Either of the following:

- Ia: No grain of the kind for which the storage problem is being considered, or a substitute therefor, is imported or exported.
- Ib: Imports and exports are known in advance (predictable with certainty and independent of the amount of storage). Ib of course, includes Ia as a special case.

*Restriction II.*—Any one of the following:

- IIa: The cost of transporting the grain within the nation is zero.
- IIb: All of the storage for the grain is located at a single point in the nation, or within a single region within which transport costs for the grain are zero.
- IIc: All of the grain (1) is produced at a single point or within a single zero-transport-cost region and (2) is consumed at the same or a different single point or within a single zero-transport-cost region.

Although these restrictive conditions are never completely satisfied in the real world, they may be approximately satisfied for certain grains. If so, application of the results given in the first section should give a storage policy that is a reasonably good first approximation to the optimal—at least a better approximation than is possible, except by chance, by the use of other existing techniques. Approximate satisfaction of restriction I, for example, means that unpredictable

<sup>4</sup> For further comment on this point, and some illustrative data, see Appendix note 4.

fluctuations in exports and imports are small, relative to total production or consumption of the grain. Similarly, restriction II is approximately satisfied if all but a small portion of the grain is stored in one small subregion, or if the relevant cost of transport is small relative to the sum of storage cost plus interest charges. For all feed grains in the United States, for example, fluctuations in net imports in recent years typically have been between 0 and 2 percent of total domestic production; and, though production, utilization, and storage occur throughout the country, they tend to be concentrated in the North Central States, where, for example, more than 80 percent of total October 1 stocks of corn are typically held.

Furthermore, biases in the computed storage rules that are caused by assuming that both restrictions I and II are true, when, in fact, they are not, are in opposite directions, so that they at least partially offset each other. That is, the assumption of restriction I results in rules which prescribe "too much" storage, since holding exports and imports constant means that effective demand for the domestically-produced grain is less elastic than it would otherwise be; whereas the assumption of restriction II results in rules that typically prescribe "too little" storage.

With all the above considerations in mind, a direct application of the analysis of this section to the storage problem for total feed grains in the United States should give a fairly close approximation to optimal storage policy; accordingly, the empirical applications are made to those grains.

Finally, it should be mentioned that, while a complete solution of the multiregional storage problem involves a formidable computational and empirical complexity, relaxation of restriction I can be allowed for with only a relatively minor modification of the "model," provided adequate empirical information is available about foreign demand, supply, and storage policy.<sup>5</sup>

Naturally, the approach developed here is equally applicable to a commodity produced and consumed within a smaller self-contained region.

### CONDITIONS USED IN DEVELOPING AND APPLYING THE RULES

As already indicated, the conditions which are relevant and which must be estimated prior to the derivation of storage rules are the following: (1) A discount factor which equals  $1/(1+r)$ , where  $r$  is the interest rate. This is the present value of one dollar due the following year, and reflects the fact that whenever commodities are held in storage, an amount of capital equivalent in value is unavailable for investment elsewhere. (2) The direct cost in dollars of carrying over the quantity stored for one year. Naturally, this total depends on the quantity stored, though certain fixed costs regardless of quantities also may be involved. (3) The total value, measured in dollars, attributable to the use of the variable quantity available for consumption ( $Y$ )

<sup>5</sup> A solution that incorporates foreign trade was obtained and applied to compute national aggregate storage rules for wheat in an unpublished manuscript entitled "The Storage of Grains to Offset Fluctuations in Yields" by R. L. Gustafson. The general approach is summarized in Note 12 in the Appendix.

in that year. (4) The probability distribution of yields per acre, since we have specified that the acreage is known.

In some cases it may be more convenient and more illuminating to use a marginal value function instead of the total value function. For readers unacquainted with calculus, we note that the marginal value function, or first derivative of the total value function (assuming that the derivative exists), is somewhat analogous to, and in some cases may be taken as identical with, a market demand curve, properly defined. (See pages 13-15.) Mathematical derivations of optimal storage rules for each of these value functions are given, but our initial presentation of the solution is in terms of the total value function, as the exposition and proofs are more straightforward in those terms.

The meaning of each of these conditions, and problems involved in measuring them empirically, are discussed in the following paragraphs.

*The discount factor.*—The discount factor is simply a transformed expression for the interest rate; but the question arises, What is "the" appropriate interest rate to use? In a free capital market, the appropriate value is the rate of return that the capital resources used in the storage program could earn in alternative uses, so the problem is to determine or estimate what that rate is. In a situation that involves capital rationing, the problem may become more complex, but we cannot here go into all the issues involved. One necessary restriction, to make the solution feasible, is that the annual discount factor be less than unity (that is, the interest rate be greater than zero). In the section on applications, we assume a range of possible values of the discount factor to obtain an estimate of the effect of such variation on the resulting optimal storage rules.

*Storage costs.*—The cost of storage is here taken essentially to be the amount of money it costs to store a given quantity of grain for a year. Serious problems of estimation are involved, however, as costs vary considerably in different locations and in different types of storage facilities, and a national aggregate is desired. The approach taken is to assume a range of possible cost estimates in order to show the effect of variation of this sort on the storage rules.

A question may arise as to whether the money cost of storage is by itself an adequate measure of the actual net cost to the economy of having a certain quantity on hand at a given time. For example, a "convenience benefit" may accrue from the existence of the stocks themselves which, if it exists, should be subtracted from the money cost of storage to obtain the actual net cost. The possibility of such a convenience benefit may be explained as follows:

It has been observed that when stocks of grain on hand are low, farmers and processors sometimes hold grain for use at a future date even though they know (via the futures market) that they could obtain similar grain at the future date at a cost less than the current value of what they hold. The resulting monetary loss, as it is incurred voluntarily, is presumably offset by a convenience benefit accruing from the holding of the grain. [See Working (14).] If benefits to the general public correspond to these private convenience benefits, and if they could be suitably aggregated, then the resulting total convenience benefit should be subtracted from the money cost of storage to obtain the net cost of storage. It is possible that, by this

adjustment, the cost of storage, for low levels of stocks, would be considerably altered. In the empirical applications given in this bulletin, we do not attempt to estimate these concepts, but rather follow the simple expedient of taking as given a fixed level of minimum working stocks, below which the carryover is assumed never to fall. The computed storage rules, then, refer to quantities of carryover above the minimum working stocks. This procedure is equivalent to assuming that when stocks fall below a certain level the convenience benefit of stocks on hand becomes indefinitely large, whereas for stocks above this level, the added convenience benefit is negligible.

It should be noted that the "convenience benefit" being discussed here is conceptually quite separate from and independent of the gain to the general public concept defined on p. 17 *et seq.* The former accrues from the existence of the stocks themselves, whereas the latter arises from the year-to-year adjustment (by means of the storage rules) of quantities utilized in accordance with changes in supply and demand conditions.

*The total value function.*—This states the value in dollars, to the general public as a whole, of utilizing the quantity  $Y$  of the grain in the year  $t$ . The problems involved in the statistical determination of a value (or utility) function of this sort from market data are highly complex, and a completely rigorous solution, applicable to the real world, probably is impossible. Nevertheless, if *any* storage policy is to be adopted, *some* value function must be decided upon before the policy can be justified or made rational. In other words, before we (that is, the general public) can decide how best to distribute quantities of the grain utilized among years, we must decide what is the value to us of utilizing alternative quantities in each of the years. Some degree of arbitrariness or statistical approximation may be inevitable, but a policy which is based on even an approximate value function is certainly likely to be better than one which ignores the problem of evaluation. Furthermore, by making use of alternative explicit value functions, we can determine the effects on storage policy of making alternative choices, or of errors in the estimate, of the value function.

In the paragraphs following, we give what appears to be the most practicable way of objectively determining, at least approximately, a function which states the value in dollars, to the general public as a whole, of utilizing a given quantity of grain in a given year. But it should be emphasized that the method of solving the storage problem which is discussed later does not depend on this particular choice of a value function, but is sufficiently general to permit the incorporation of a wide variety of possible functions. For example, if the Government should decide that the storage program should be operated so as to maximize the expected total revenues of grain producers, we could, by simply setting "total value" equal to "total revenues of grain producers" in our solution, obtain storage decision rules which would be "optimal" in that sense.

We define the suggested total value function by first defining a particular kind of market demand curve, or market price-quantity relation, for the grain, as follows: The quantity of resources used in the production of the grain are assumed given (constant), but the quantity of grain produced varies from year to year, owing to purely

noneconomic forces, in particular, the weather. The grain produced, whatever the quantity, is thrown on the market, and the maximum price is determined at which that entire quantity can be sold and consumed. That is, no year-to-year carryover is allowed. For all points along the resulting price-quantity relation, the total productive capacity of the economy, *except* for the quantity of the grain which becomes available, is assumed given (constant); and the price level of all goods and services other than the grain is also held constant. However, allocations of *particular* other resources and *relative* prices of other goods and services are *not* assumed to be fixed, but are allowed to shift in response to the changes in the quantity of the grain, to the extent that the market equilibrating forces in the economy do in fact cause them to shift within the crop year.<sup>6</sup>

The resulting price-quantity relation is defined as the marginal value function for the grain; it gives the per-unit value, in terms of other goods and services, which the general public, operating through the market, places on the grain when the total quantity is  $Y$ . By this definition, we essentially make this value not directly dependent on the income redistribution effects of the changes in the grain supply. This appears to be the most feasible procedure, the alternative being to adopt arbitrarily some interpersonal or intersectoral weighting, such as would be implied by setting total value equal to total revenue of grain producers.

Total value then can be defined as the area under the marginal value (or demand) curve between  $O$  and  $Y$ . However, in most cases some quantity, which can be taken as a constant, exists below which the quantity utilized never falls. Conceptually, this quantity may be close to zero for items that are relatively unessential in the diet of either human beings or animals, and considerably above zero for dietary essentials with few substitutes. Alternatively, we may look on the existence of this minimum quantity as simply an empirically observed fact. Since we never can obtain observations regarding the nature of the total value when the quantity is below this minimum, we take these values as unknown constants which can be conveniently ignored since, in the maximization process by which the optimal storage rules are obtained, they have no effect on the results. *We may, therefore, define the total value function as "the increase in real national (or regional) income which is attributable to increasing the amount utilized of the grain from the minimum value of  $Y$  ( $Y_{min}$ ) to  $Y$  itself, when other productive capacity is given.* That is, total value is the area under the marginal value curve between this minimum and  $Y$ .<sup>7</sup> The total value can be thought of as the value of other goods and services

<sup>6</sup> Note that the demand curve so defined differs slightly from the usual definitions of the (Marshallian) demand curve in that we hold neither real income nor money income and other prices constant along the curve. The demand curve also is defined for a relatively "short run," and hence tends to be less elastic than a long-run demand curve.

<sup>7</sup> This is readily seen by considering the effect on real income of a small change in the quantity of the grain, say  $dz$ , from an initial quantity  $z$ ; the resulting change in real income is the change in quantity times the per-unit value. Adding up these small changes in real income between  $Y_{min}$  and  $Y$  gives the total value. In economic literature, this value frequently is referred to as "total social value." It should be noted that "total value," as used here, does *not* mean "total revenue," or price times quantity consumed; it is, rather, the entire area under the marginal value function.

which society is willing to give up in order to utilize the quantity  $Y$  rather than the minimum quantity.

*An empirical estimate of the marginal value function for feed grains.*—We next consider the problem of empirically estimating the marginal value function for all feed grains. We allow approximately for effects of changes in other productive capacity and the price level in the usual way, namely, by including appropriate income and price indexes in the estimation model. The main difficulty in the case of the feed grains arises from the facts that (1) an important factor in determining within-year demand for feed grains is the beginning-of-year level of livestock inventories on farms, so that to estimate the within-year price function it is necessary to include this variable, which may, for this purpose, be treated as predetermined; but (2) an important effect of a change in a given year's supply of feed grains is to change the following year's livestock inventories, so that to determine the total effects of year-to-year changes in the grain supply, such effects on livestock inventories should be taken into account.

For example, the 5-equation model of the feed-livestock economy developed by Hildreth and Jarrett (5), using their limited information estimates of the coefficients, indicates an elasticity of livestock products produced with respect to quantity of feed grains fed of 0.22. However, if their five equations are reduced to a single one for which quantity of livestock products sold is made a function of quantity of feed grains fed, quantity of protein feeds fed, and the predetermined variables, the resulting elasticity of livestock products sold with respect to feed grains fed is  $-0.03$ , which does not differ significantly from zero. A similar result is obtained from the 4-equation model developed by Foote (3). The difference between 0.22 and  $-0.03$  (or zero) presumably represents the effect on the following year's livestock inventories of a change in a given year's quantity of feed grains fed. The corresponding coefficients, using the Hildreth-Jarrett least squares estimates, are 0.35 and 0.14, respectively.

If we take these year-to-year adjustments in livestock inventories into account, the price of feed grains in a given year is a function not only of the quantity utilized in the given year but also of the quantities utilized in preceding years. Using again the Hildreth-Jarrett model, and holding the predetermined variables (except livestock inventories) constant, one can determine the net effect, taking into account the resulting changes in the other endogenous variables, of a change in a given year's quantity of feed grains fed on the following year's price of feed grains. Using the limited information estimates, the result is:

$$\log P_t = -1.47 \log Y_t + 0.43 \log Y_{t-1} \quad (5)$$

where  $P_t$  is the price of feed grains in year  $t$  and  $Y_t$  is the quantity of feed grains fed in year  $t$ . The lag effect actually extends back for more than one year, of course, but for our purposes consideration of the 1-year lag is sufficient. The least squares estimates of the coefficients give:

$$\log P_t = -1.51 \log Y_t + 0.43 \log Y_{t-1} \quad (5.1)$$

These results seem to indicate that one ought to make the marginal value a function of lagged quantity utilized as well as current quantity

utilized. This can be done fairly readily in a formal solution to the storage problem, but the resulting computational requirements become much greater, and the resulting storage rules more complicated. The problem may be stated as follows: What we actually have is a function of the form

$$P_t = a_0 Y_t^{-a_1} Y_{t-1}^{a_2} \quad (5.2)$$

whereas we would like to have, if possible, a function of the form

$$P_t = b_0 Y_t^{-b_1} \quad (6)$$

which, for purposes of storage policy, is equivalent, or at least approximately so, to what we actually have. Fortunately, an equivalent function can be obtained since, for purposes of storage policy, we are concerned with the interrelationships among  $P$  and  $Y$  in successive years, that is, among, say,  $P_{t+1}$ ,  $P_t$ ,  $Y_{t+1}$ , and  $Y_t$ .<sup>8</sup>

<sup>8</sup> The truth of this is demonstrated by considering the following two sets of relations:

$$\text{I: } P_{t+1} = a_0 Y_{t+1}^{-a_1} Y_t^{a_2}$$

$$P_t = a_0 Y_t^{-a_1} Y_{t-1}^{a_2}$$

$$\text{II: } P_{t+1} = b_0 Y_{t+1}^{-b_1}$$

$$P_t = b_0 Y_t^{-b_1}$$

Take the ratio of  $P_{t+1}$  to  $P_t$  in each case, giving, say,  $R_I$  and  $R_{II}$  respectively. Then take the elasticity of this ratio with respect to  $Y_t$  in each case, giving respectively:

$$\text{I: } a_1(1-e) + a_2$$

$$\text{II: } b_1(1-e)$$

where  $e$  is the elasticity of  $Y_{t+1}$  with respect to  $Y_t$ , that is, the percent change in  $Y_{t+1}$  which occurs as a result of decreasing the carryover in year  $t$  by one percent of  $Y_t$ . It follows that the elasticity of the ratio  $R$  with respect to  $Y_t$  is the same in cases I and II if

$$b_1 = a_1 + \frac{a_2}{1-e}$$

The value of  $e$  depends on the values of  $Y_{t+1}$  and  $Y_t$ , and on the storage rule to be applied in year  $t+1$ , but the value of  $e$  is always negative or zero. Hence, values of  $b_1$  which make II approximately equivalent to I for storage policy purposes are given by

$$a_1 \leq b_1 \leq a_1 + a_2$$

For all feed grains in the United States, it can be shown that

$$0 \geq e > -2$$

so that

$$a_1 + \frac{a_2}{3} < b_1 \leq a_1 + a_2$$

Based on the Hildreth-Jarrett limited information coefficients,

$$1.62 < b_1 \leq 1.90$$

*Probability distribution of output.*—The probability distribution of output is estimated from observations on the variation in yield per acre in past years, making due allowance for trends. Such data are available from the records of the Crop Reporting Board for major crops back to 1866. Ideally, provision would be incorporated to allow for the way in which year-to-year variations in acreage planted are determined in a free market by the interrelationship of supply and demand factors. As little definite information is at present available about supply functions for grains, this refinement has not been made. If better information on the economic determinants of acreage planted become available, such knowledge can and should be incorporated into the solution directly. In the meantime, the results obtained may be regarded as first approximations, the adequacy of which depends on the accuracy with which acreage planted in future years can be predicted. A further justification for initially emphasizing the fluctuations in yield per acre and neglecting variations in acreage planted is that, except in years for which acreage allotments are in effect, the major proportion of the variation in year-to-year output is due to variations in yield. A final justification is that acreage for many crops can be controlled or predicted, whereas yields cannot, and a storage program of the sort being considered here can be looked on primarily as a policy designed to mitigate the economic effects of noncontrollable and nonpredictable fluctuations.

### DEFINITION OF AN OPTIMAL STORAGE POLICY

Having defined and briefly explained the conditions used in deriving the storage rules, we now proceed to define the criterion of optimality which the rules are intended to satisfy. First, the gain incurred in any given year, that is, the year  $t$ , is defined as the total value of the grain utilized minus the cost of storage for grain to be carried into the next year. Some readers may feel that, in the definition of gain, the total value of the grain that would be utilized in the absence of any storage should be subtracted out. But the effect of this change in the definition is simply to introduce a set of constants into the system, a condition that has no effect on the maximization process by which we obtain the storage rules. That is, if the latter concept is thought of as a "net gain," the storage rules that maximize total gain are identical to those that maximize net gain. From a mathematical standpoint, it is easier to work with the simpler concept of total gain.

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Footnote 8—Continued

Using the least squares coefficients,

$$1.66 < b_1 \leq 1.94$$

Coefficients like  $b_1$  are referred to as "flexibilities." That is, the flexibility of the marginal value function is the absolute value of the elasticity of marginal value with respect to quantity utilized. We mainly are concerned with determining an "upper limit" estimate of the flexibility of the marginal value function since, as might be expected, the higher the flexibility, the higher are the resulting optimal storage rules and storage levels.

Use of the term "flexibility" is convenient to emphasize that in this context quantity utilized is treated as the independent variable and marginal value or price as the dependent, rather than vice versa. That is, the flexibility of a price function is the inverse of the absolute value of the "elasticity" of the same function treated as a demand curve.

We now wish to consider the factors that determine the total gain in any year. The quantity utilized depends on (1) the initial carryover from the previous year, (2) production in the current year, and (3) the carryout in the particular year. But if we consistently follow a set of storage rules, the carryout depends on the particular rule that is adopted. Since, with any given marginal value function, the total value depends on the quantity utilized, this value in turn depends on the initial supply, which is equal to the initial carryover from the preceding year plus production in the current year, and the storage rule. Likewise, total storage cost, for any given level of interest rates and cost per unit stored, depends on the amount stored. Thus, the total gain from storage depends on the initial supply and our storage rule.

In thinking about some year in the future, production cannot be estimated in advance but depends on the particular yield that happens to prevail. In connection with variables of this sort, in cases where it is felt that the variable can be treated as though subject to a probability distribution that is known or can be estimated, statisticians use a concept known as an "expected value." To take an example, the expected yield, in this sense, equals the sum obtained by multiplying each possible yield by its probability of occurrence, and adding the resulting products.

Considering any given future year, then, we can think of applying a given storage rule to each possible total supply in that year. The total supply depends, of course, on the carryover from the preceding year, acreage planted, and yield. The total gain from storage can then be computed for each possible total supply, or equivalently, taking acreage as given, for each possible carryover from the preceding year and each possible yield. Next, for each possible carryover from the preceding year, the "expected gain" in the given year is obtained by multiplying the gain corresponding to each possible yield by the probability of occurrence of that yield, and adding the resulting products. Thus, the expected gain in any given future year, under given conditions, depends on the storage rule applied in that year, and on the carryover from the preceding year. Of course, the carryover from the preceding year depends on the total supply and the storage rule applied in *that* year, and so on back to the current year. It should also be noted that the expected gain in any given future year is *not*, in general, equal to the gain that would be computed by applying the given storage rule to the expected *yield* (or expected supply) in that year.

We now define a new variable: The gain in the current year plus the sum of expected gains for all relevant years in the future discounted back to the current year. The size of this variable, under given conditions, depends only on the supply in the current year and the particular set of storage rules being applied. Finally, we define the optimal storage policy as that set of storage rules that maximizes this sum of discounted expected gains, for any given initial supply.

In the paragraphs that follow, total values that relate to all possible levels of utilization are referred to collectively as the total value function, and costs of storage that relate to all possible levels of storage are referred to collectively as the cost of storage function. The term "function" carries the same connotation when used elsewhere. Mak-

ing use of this concept, the criterion of optimality is the following: Given (1) the probability distribution of yields, (2) the total value functions, and (3) the cost of storage functions for an  $n$ -year period ( $n \geq 2$ ), the optimal storage policy for the period is defined as that set of storage rules which maximizes the sum of discounted expected gains over the period, where the gain for each year is the total value of the quantity utilized minus the cost of storing the amount carried over.

## METHOD OF SOLUTION

The solution to the grain storage problem presented here is an adaptation of a solution to an inventory problem developed by Dvoretzky, Kiefer, and Wolfowitz (2). Some reformulation of the framework and proofs was required to adapt them to the grain storage problem. In the inventory problem, the administrator at the beginning of each period is given an initial stock, a cost of ordering function, a cost of storage function, a probability distribution of demand (which may be based on historic data), and a penalty function which depends on the quantity of unfulfilled demand; he must decide how much to *order*. In the grain storage problem the important random variables are future harvests, and the decision is taken with respect to how much of currently available supply should be currently utilized and how much carried over for future use. Another difference, which may be important in applications in that it leads to troublesome discontinuities, is that in the grain storage case, unlike what may be possible in the inventory case, we must exclude the possible existence of negative carryovers.<sup>9</sup> Finally, it may be mentioned that the mathematical development pertaining to use of the marginal value function (pages 44-59 and related Appendix notes) is original here.

We start by introducing what is perhaps the crucial aspect of the proposed solution, that is, the device which permits us to avoid (1) the necessity of assuming in advance anything about the forms of the storage rules and (2) the laborious computation of expected values as functions of the parameters of the forms adopted. We have seen that the determination of the optimal rule for any year depends on the rules to be followed in succeeding years. Hence the only way to avoid making assumptions about rules in succeeding years is to start with the year that has no succeeding year, namely, the last year of the period, and work backward. We do just that.

Under certain conditions, the length of the relevant period, that is, the "time horizon," must be assumed in advance. However, in cases

<sup>9</sup> The interested reader also should refer to Arrow, Harris, and Marschak (1); these authors chronologically preceded, and laid the conceptual groundwork for, the work of Dvoretzky, Kiefer, and Wolfowitz (2), and in their work the concepts of a utility (or penalty) function and of a controlled stochastic process were for the first time introduced into the English-language literature on the inventory problem.

The mathematical formulation of the problem and its solution, as presented in this bulletin, are intended to be complete and sufficient for our purposes. Our presentation is more elementary than that of Dvoretzky et al. (2), and the results thereby lack some generality. However, some of the ways in which the solutions may be generalized are indicated in later sections and in the Appendix, and the reader, once he understands the basic concepts involved, should be able to provide the modifications required for any particular application.

where the relevant conditions, that is, the value function, the cost of storage function, the interest rate, and the probability distribution of output, can be assumed to be the same in each future year, it is not necessary to make any assumption about the relevant number of years. The relevant criterion of optimality for these cases is the maximization of the sum of discounted expected gains in *all* future years, and, as might be expected, it turns out that the optimal storage rule is identical in each year. The mathematical method then itself converges to this single optimal rule which is applicable every year.

To illustrate the procedure, we first consider the general case, that is, the one for which the storage rules can vary from year to year. (1) We first determine the rule for the  $n$ th or last year in the following way: For all possible total supplies at the start of the year, we find that carryout that maximizes the gain. This is our storage rule for that year. Given the rule, the maximum gain depends only on the size of the initial supply. (2) We now make use of the statistical concept of an "expected" gain (see page 18). The expected maximum gain depends only on the size of the initial carryin, since we multiply all possible levels of production by their respective probabilities of occurrence. The initial carryin for the  $n$ th year is the same as the carryout for the year  $n-1$ . For every possible level of supply in the year  $n-1$ , we find that carryout that will maximize the *sum* of the gain in that year *and* the discounted maximized expected gain in year  $n$ . By the same reasoning as used previously, the expected maximum value of this figure depends only on the size of the carryout in the year  $n-2$ . (3) Using the same procedure, we continue back to year 1, whereupon we have determined a set of storage rules, one for each year, which maximize the sum of discounted expected gains for the entire period.

Cases where the value function, cost of storage function, interest rate, and probability distribution of output can be assumed to be the same in each future year are called cases of "stationarity." In such cases, as already indicated, the optimal storage rule is also the same in each year. This single optimal rule can be shown to be the unique solution of a single equation.<sup>10</sup> The computations required to *obtain* the solution, however, are, at least in the general case, of the iterative type analogous to those used for cases of nonstationarity. The main difference is that, in cases of stationarity, the iterations are continued until convergence is achieved.

Such an assumption of stationarity is not as restrictive or unrealistic as might at first appear. For computational purposes, we assume that the conditions are unchanging in all future years. But the optimality of the resulting rule, as applied to the current year only, does not require that the conditions in fact remain unchanged in all future years; all that is really required is that the same storage rule applies in the *next* succeeding year. Such a condition is satisfied if, for example, the storage rule for the next succeeding year is also calculated assuming stationarity and using the same estimates of the conditions as are used this year. Of course, if it is known in advance how the conditions will change in future years, such knowledge should be incorporated directly into the solution.

<sup>10</sup> For mathematical proofs, see pages 40-47 and 74-80.

Complete mathematical solutions, using both the total and the marginal value functions, are given beginning on page 40. These are followed by a discussion of some special mathematical relationships of interest to the economic analyst. As these sections require a rather advanced knowledge of calculus, we first show the results of applying these methods to obtain storage rules for feed grains, then summarize some general conclusions with respect to storage that can be developed by examining the mathematical nature of the rules, and finally show a method of obtaining approximations to the rules that requires only a use of direct arithmetic operations.

## APPLICATIONS TO FEED GRAINS

In this section results of some computations of optimal storage rules for "corn equivalents" of aggregate feed grains for the United States are shown. The feed grains are here taken to be corn, oats, and barley. Sorghum grains were omitted because of a lack of adequate information on acreage planted for grain. Ideally, sorghum grains used for feed should be included, but the effect on the final results would be negligible, as production of sorghum grains in the United States averages about 3 percent of the production of total feed grains. Bushels of oats and barley were converted into corn equivalents on the basis of their respective relative number of pounds of digestible nutrients per bushel, as follows:

Grain	Corn equivalent of one bushel <sup>1</sup>
Corn.....	1. 000
Oats.....	. 488
Barley.....	. 806

<sup>1</sup> Slightly different corn equivalents are being used currently by the United States Department of Agriculture.

The total supply of corn equivalents in each year was obtained by converting the supply of each grain, in bushels, into corn equivalents, in accordance with the above ratios, and adding. A different set of conversion factors should perhaps have been used for that part of the grain used for purposes other than as a livestock feed.

In determining storage rules for two or more grains simultaneously, it would be theoretically preferable to set up a model incorporating explicitly both economic substitution relations and joint probability distributions of output, rather than to use fixed ratios of substitution as done here. The formal solution for such a model is analogous to the solution for the multi-regional problem discussed on page 60, with a comparable increase in computational difficulties. Another problem is that of empirically estimating the substitution relations. Analyses by Foote (3) and Meinken (7) indicate that the price-elasticity of demand for corn alone, holding quantity of other feed grains fed constant, is not significantly different from the price-elasticity of demand for all feed grains. The corn equivalence ratios used here are roughly equal to the average price ratios between the grains in recent years.

All applications assume stationarity (see page 20), and independence. Since acreage planted is thus assumed constant, the data and computations were made more manageable by taking all quantities on a per acre basis. Thus the probability distribution of output is the probability distribution of yield in bushels per planted acre; total supply, quantity utilized, and carryover are in bushels per acre; and marginal value and marginal cost of storage are in dollars per bushel per acre. All of these quantities can be translated into approximate national aggregates by multiplying by 140 million acres, the approximate average number of acres planted to corn, oats, and barley in recent years.

The probability distribution of yields was estimated from records of the Crop Reporting Board of the actual variability of yields in the period 1901-1950 as follows: For each year, total production of each grain was converted to its corn equivalent, and the result was added to get the corn equivalent of aggregate production of feed grains. This figure was divided by the total acreage planted to corn, oats, and barley in the given year to get the aggregate corn equivalent yield per acre.<sup>11</sup> A 5-year moving average of a 9-year moving average was fitted to the resulting yields, omitting the drought years 1934 and 1936, to obtain an estimate of the trend. If  $x_t$  is the actual yield in year  $t$  and  $T_t$ , the trend value for that year, we let

$$d_t = x_t - T_t \quad (7)$$

and

$$z_t = 30 + (\frac{1}{2}) [d_t + (d_t/T_t)30] \quad (8)$$

where 30 was an estimated yield for 1954. Thus, to estimate the variability of yields in future years, an arithmetic average of the actual and the relative deviation from trend in past years was used. This assumption is conservative, that is, it probably gives a higher estimated variability than may actually occur, since the trend yield has increased substantially over the period. The resulting  $z_t$ 's were then grouped into one-bushel intervals centered on integers, giving the following distribution:

Yield per planted acre		Relative frequency $f(x)$	Yield per planted acre		Relative frequency $f(x)$
Range	Midpoint $x$		Range	Midpoint $x$	
<i>Bushels</i>	<i>Bushels</i>		<i>Bushels</i>	<i>Bushels</i>	
18.5-19.5----	19	0.02	27.5-28.5----	28	0.06
19.5-20.5----	20	.02	28.5-29.5----	29	.22
20.5-21.5----	21	.00	29.5-30.5----	30	.20
21.5-22.5----	22	.02	30.5-31.5----	31	.14
22.5-23.5----	23	.00	31.5-32.5----	32	.10
23.5-24.5----	24	.00	32.5-33.5----	33	.10
24.5-25.5----	25	.02	33.5-34.5----	34	.00
25.5-26.5----	26	.02	34.5-35.5----	35	.02
26.5-27.5----	27	.06	-----	-----	-----

<sup>11</sup> For 1901-1928, acres planted for each grain were estimated by multiplying acres harvested by the weighted average ratio of acres planted to acres harvested in 1929-50.

The mean of the distribution is 29.46 bushels per acre; the standard deviation is 3.03 bushels per acre. As is typical for yields of crops, the distribution is skewed to the left.

The assumption of stationarity in computation of storage rules so far as yields are concerned may be looked upon as an assumption that average or normal yield per acre in future years will be 29.46 bushels. The discrepancy between 29.46 and 30 is mainly due to the omission of the drought years 1934 and 1936 in obtaining the trend. A sharp upward movement in the trend yield started about 1933, reflecting in part the introduction of hybrid seed corn. Since use of hybrid seed reached almost 100 percent within the main areas of production by 1950, it seemed reasonable to assume, when this study was begun, that the trend would level out at something slightly above the average yield for corn equivalents of feed grains from 1949 through 1955 of 28.7 bushels. A continued upward trend in more recent years is believed to reflect progress in techniques of production and increased use of fertilizer, irrigation, and other inputs of this sort. In applying the storage rules, an upward trend of this kind could be allowed for by changing the assumed mean yield from time to time while retaining as a measurement of variation around the mean the long-term historic pattern based on deviations from trend.

One way to justify an assumption of stationarity in the computations is to assume that any future trend in supply (acreage or yield) will be partially in response to, and partially offset by, the trend in demand, so that the resulting trend in the real price of the grains will be small enough to be neglected for storage policy purposes. The real price of corn over the last 80 years has followed a slight upward trend, amounting to an average of roughly 0.6 percent per year. Such a trend, if assumed to continue into the future and incorporated into the computations of the storage rules, would have a relatively small effect on the results.

In all applications except one, the marginal value function is assumed to be linear. This is mainly a computational convenience, since empirical demand studies for corn and feed grains have generally shown that a linear relationship gives about as good a fit to the data as a logarithmic or constant-elasticity relationship (for example, see Foote, Klein, and Clough (4), Hildreth and Jarrett (5), and Shepherd (10), (11)). Computations in some representative cases that have used the two alternative assumptions indicate that the optimal storage rule using a logarithmic marginal value function differs little from the optimal rule using a linear marginal value function with, of course, the same estimates of the other conditions and the same average flexibility of marginal value in each case.

Results presented here are the computed optimal storage rules for aggregate feed grains, under alternative assumptions about the conditions, that is, the annual discount factor, the marginal value function, the marginal cost of storage, and the distribution of yields. The subscripts on the  $\theta$ 's designating the rules do not stand for years or iterations, but for alternative optimal (stationary) rules, applicable under the respective sets of conditions specified. (See table 1.)

The application intended to approximate conditions in an idealized free market is based on a price elasticity of demand for aggregate feed grains,  $\eta_0$ , of  $-0.50$ . This is the elasticity at  $Y=Y_0=30$  bushels per

acre. This is consistent with, though slightly conservative with respect to, the upper limit estimate of the flexibility of the marginal value function,  $\epsilon_0 = -1/\eta_0 = 1.94$ , obtained in footnote 8 on page 16 based on the Hildreth-Jarrett (5) estimates of the coefficients after allowing for an annual lag effect. It also is consistent with results reported by Foote, Klein, and Clough (4), where the elasticity of demand for "total feed grains or possibly for total feed concentrates" is estimated to be between  $-0.40$  and  $-0.50$ , based on year-to-year changes. The market price at the quantity consumed when  $Y=30$  bushels per acre was taken to be \$1.50. This gives a market price function

$$\rho(Y) = \$1.50 - \$0.10(Y-30) = \$4.50 - \$0.10(Y) \quad (9)$$

where  $Y$  is in bushels per acre and  $\rho(Y)$  is a mathematical symbol representing the marginal value function.

To determine the effects of the possible existence of losses to the general public attributable to fluctuations in utilization not measured by changes in market price, computations were carried through using marginal value functions with flexibilities of  $\epsilon_0 = 2.5$  and  $\epsilon_0 = 3.33$  (the corresponding elasticities of demand being  $\eta_0 = -0.40$  and  $\eta_0 = -0.30$  respectively).<sup>12</sup>

The corresponding marginal value functions are

$$\rho(Y) = \$1.50 - \$0.125(Y-30) = \$5.25 - \$0.125(Y) \quad (10)$$

and

$$\rho(Y) = \$1.50 - \$0.167(Y-30) = \$6.50 - \$0.167(Y) \quad (11)$$

respectively.

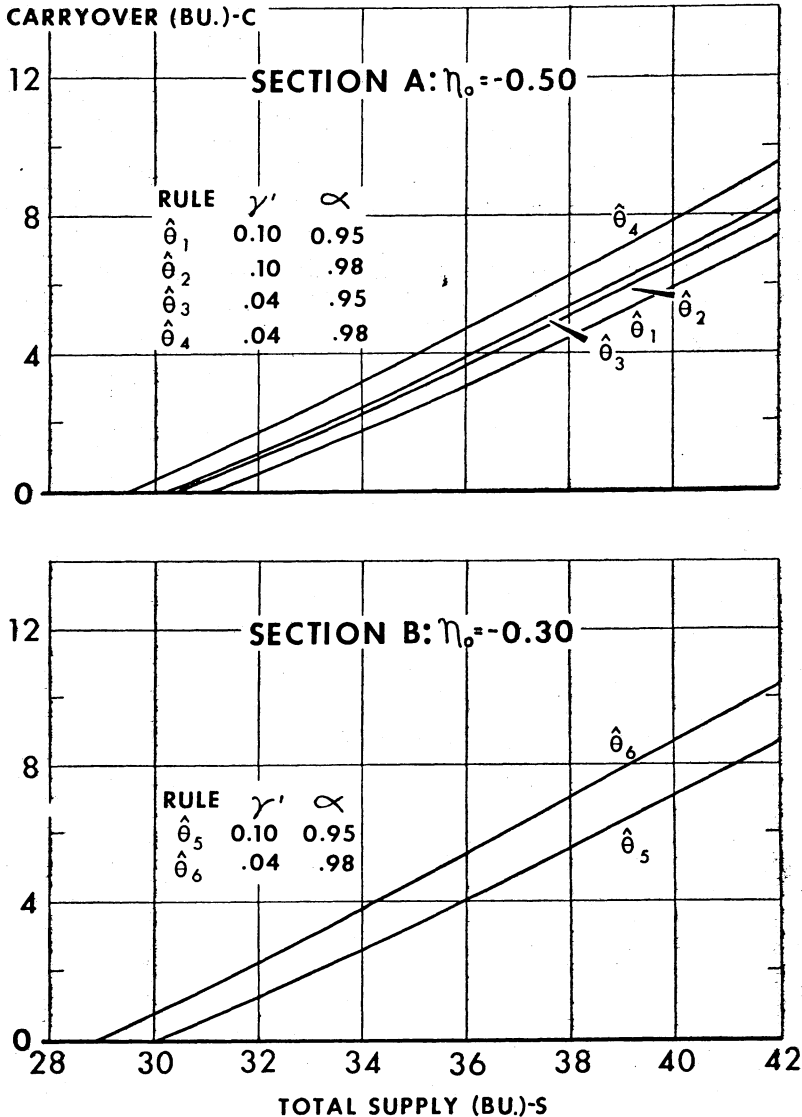
Effects on the optimal storage rule of changing the assumptions about the marginal cost of storage,  $\gamma'$ , and the annual discount factor,  $\alpha$ , also were determined. In section A of figure 1,  $\hat{\theta}_1$ ,  $\hat{\theta}_2$ ,  $\hat{\theta}_3$  and  $\hat{\theta}_4$  are optimal storage rules that result under different assumptions about  $\gamma'$  and  $\alpha$ , when  $\eta_0 = -0.50$ . The values  $\gamma' = \$0.10$ , reflecting a marginal storage cost of 10 cents per bushel per year, and  $\alpha = 0.95$ , equivalent to an interest rate of 5 percent, are estimates of the approximate actual cost of storage and discount factor, respectively, under conditions that existed in the early 1950's. As indicated in figure 1, the alternative assumptions were  $\gamma' = \$0.04$  and  $\alpha = 0.98$ , with computations made for each of the four possible combinations. In section B of figure 1, effects on the optimal storage rule of different assumptions about  $\gamma'$  and  $\alpha$  when  $\eta_0 = -0.30$  are shown.

Effects of changes in one condition in general are not the same for different values of the other conditions, as the interaction effects are fairly complicated. Hence the optimal storage rule usually must be calculated anew for each change. The calculations, however, in some cases are simplified by making use of the equivalence relations discussed beginning on page 49.

<sup>12</sup> Reference usually is made to the inverse-flexibilities, that is, to elasticities, since to most readers a direct comparison to the usual concept of price elasticity of demand probably is more meaningful.

# FEED GRAINS:\* OPTIMAL CARRYOVER RULES PER ACRE

For Specified Values of  $\gamma'$  and  $\alpha$  When  $\sigma = 3$  Bushels per Acre, and  $\rho$  is Linear



\*CORN, OATS AND BARLEY, CORN EQUIVALENT

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Figure 1.—As would be expected, optimal carryovers are larger when cost of storage and charges for interest are relatively low. (A high value for  $\alpha$  corresponds to a low interest rate.)

Section A of figure 2 shows the effects on the optimal storage rule,  $\hat{\theta}$ , of different assumptions about the elasticity of  $\rho$ ,  $\eta_0$ , when the marginal cost of storage  $\gamma'$  and discount factor  $\alpha$  are held at their approximate actual values of \$0.10 per bushel and 0.95 respectively. Here,  $\eta_0$  was taken as equal to  $-0.50$ ,  $-0.40$ , and  $-0.30$ . Section B shows the results of similar changes when the cost of storage and the discount factor are taken at  $\gamma' = \$0.04$  per bushel and  $\alpha = 0.98$ .

Section A of figure 3 indicates the effects of changing the estimate of the variance of the probability distribution of yields when the other conditions ( $\rho$ ,  $\gamma'$  and  $\alpha$ ) are fixed at their approximate free-market values ( $\eta_0 = -0.50$ ,  $\gamma' = \$0.10$  per bushel,  $\alpha = 0.95$ ).  $\hat{\theta}_1$  is the optimal rule under the estimated actual probability distribution F, with standard deviation  $\sigma_0 = 3.03$  bushels per acre;  $\hat{\theta}_8$  is the optimal rule under a probability distribution G, which has the same mean and shape as F, but for which  $\sigma_G = (5/3)\sigma_0 = 5.05$ ; <sup>13</sup> and  $\hat{\theta}_0$  is the optimal storage rule if there were no variability whatever in future yields ( $\sigma = 0$ ).<sup>14</sup> Section B presents results under similar conditions when  $\gamma' = \$0.04$  per bushel and  $\alpha = 0.98$ .

Figure 4 shows the effects of a linear assumption about the marginal social value function  $\rho$ , as compared with results if we assume  $\rho$  to have constant elasticity, where the assumptions about the other conditions,  $\gamma'$ ,  $\alpha$ ,  $\eta_0$  and F, correspond to their actual approximate values.

In figure 5, all the computed optimal rules are shown to facilitate inter-comparisons.

The optimal storage rule  $\hat{\theta}(S)$  for each set of conditions was computed over the range of values of total supply S, in bushels per acre, from 0 to 50 (50 being 1% of a normal crop), although the charts show values of S only up to 42. The computed numerical values of all the rules are given in table 1, along with the conditions applied in each case.

The equilibrium level,  $C^*$ , also is given in table 1 for each rule. An exact definition of  $C^*$  is given on page 56. However, it may be viewed as the level toward which, for any given initial carryover, the *expected* carryover in the next year tends. It also can be thought of as a sort of average level of carryover around which the yearly carryovers over a long period tend to fluctuate under the given rule  $\theta$ . It is particularly useful to enable the analyst to make rough comparisons between "average" carryover levels that result under optimal storage rules that satisfy the criteria specified in this bulletin and carryover levels recommended by other writers, or that satisfy other criteria.

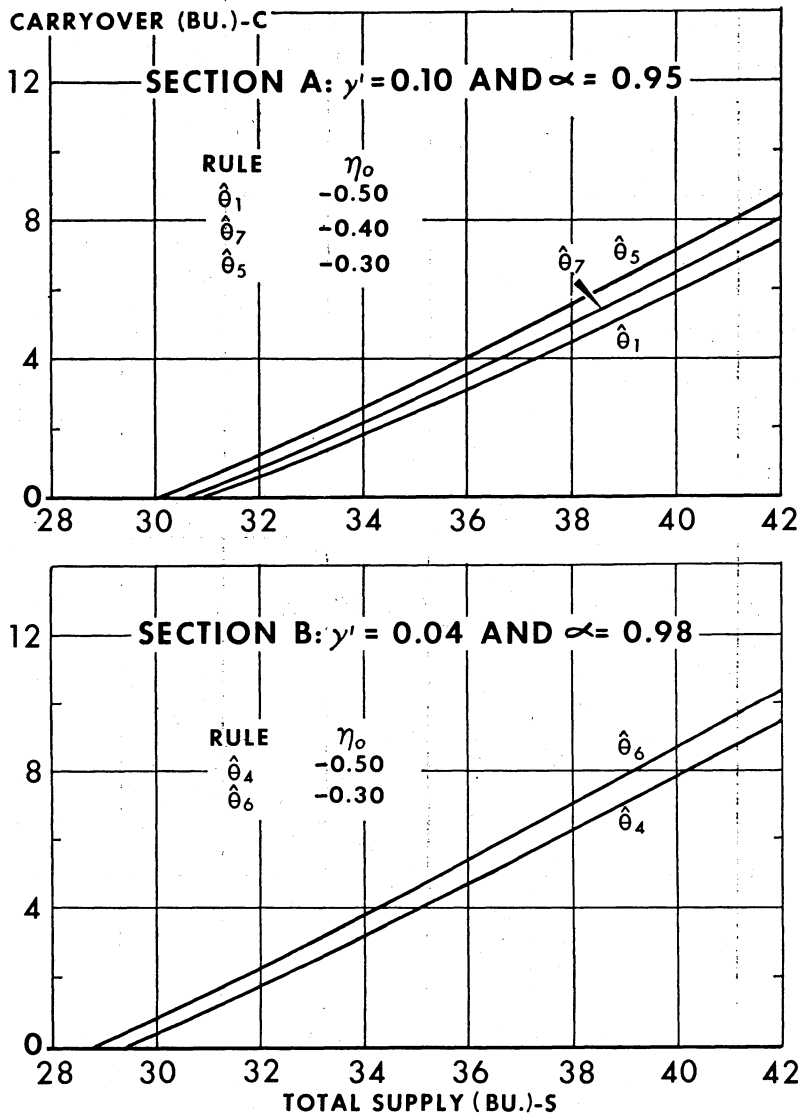
"Minimum working stocks" are the aggregate quantity of grain which farmers, dealers, processors, and so forth keep on hand to facilitate their day-to-day operations, no matter how small the total available supply. All carryovers shown here are quantities in excess of minimum working stocks, and the latter should be added to the amounts indicated if a total figure is desired. For corn equivalents

<sup>13</sup> The factor 5/3 was chosen mainly for computational convenience in using the equivalence relations discussed on p. 50, together with other computations using F. This avoided the necessity of actually computing G and carrying out a solution independently with integrations over G.

<sup>14</sup> This is essentially the case discussed by Williams (13).

## FEED GRAINS\*: OPTIMAL CARRYOVER RULES PER ACRE

For Specified Values of  $\eta_o$  When  $\sigma = 3$  Bushels per Acre, and  $\rho$  is Linear



\*CORN, OATS, AND BARLEY, CORN EQUIVALENT.

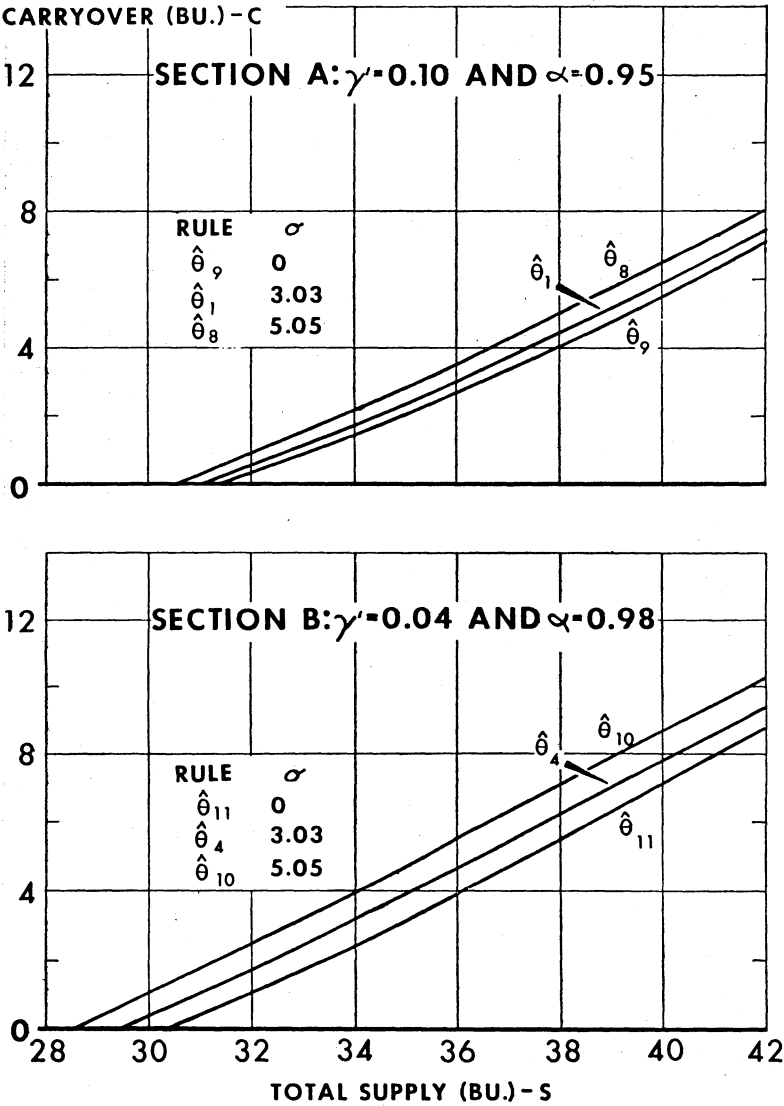
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Figure 2.—When the marginal value function is extremely inelastic, as for  $\hat{\theta}_6$ , the optimal carryover is larger than when it is less inelastic.

# FEED GRAINS:\* OPTIMAL CARRYOVER RULES PER ACRE

For Specified Values of  $\sigma$  When  $\eta_0 = -0.50$  and  $\rho$  is Linear



\* CORN, OATS AND BARLEY, CORN EQUIVALENT.  
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Figure 3.—As would be expected, the greater the variability in expected production, the larger is the optimal carryover. A comparison of the relative distance between  $\hat{\theta}_8$  and  $\hat{\theta}_1$  in section A and between  $\hat{\theta}_{10}$  and  $\hat{\theta}_{11}$  in section B indicates that the expected variability in production has a greater effect on the optimal rule when storage costs and interest rates are relatively low, as in section B, than when they are relatively high, as in section A.

## FEED GRAINS\*: OPTIMAL CARRYOVER RULES PER ACRE

When  $\rho$  is Linear or Curvilinear and  $\gamma' = 0.10$ ,  $\alpha = 0.95$ ,  $\eta_0 = -0.50$ ,  
and  $\sigma = 3$  Bushels per Acre

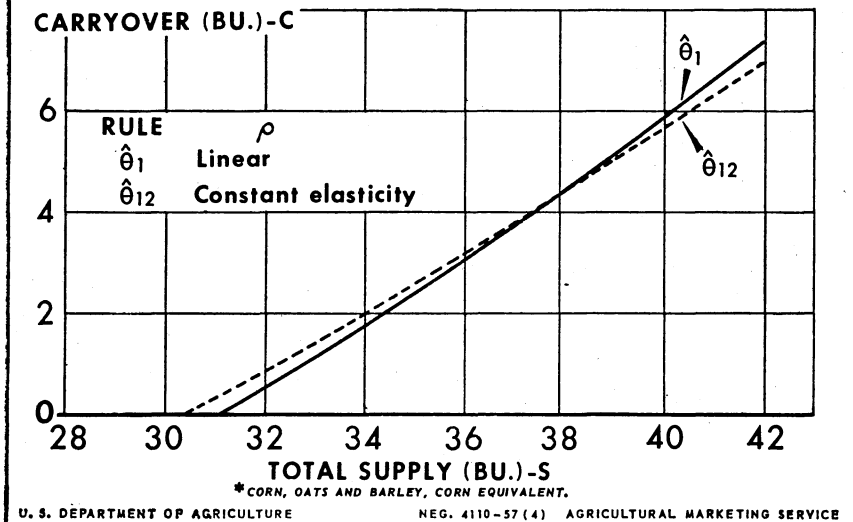


Figure 4.—When the marginal value function is of constant elasticity, the optimal carryover for small supply is higher and for large supply is lower than under similar conditions when the function is linear. In each case, however, the storage rule is curvilinear.

of all feed grains, minimum working stocks in this study were taken to be about 200 million bushels, or about 1.4 bushels per acre.

This value may be 50 to 100 million bushels lower than a minimum that would provide a reserve until quality grain is available from the next crop. The national aggregate equilibrium level including working stocks for each storage rule also is given in table 1. This is obtained by multiplying the per acre value for  $C^*$  by 140 million acres and adding to the result 200 million bushels. To convey some idea of the possible range or variability of carryovers under each rule, a value  $C^{**}$  is given for all rules except  $\theta_9$  and  $\theta_{11}$ , where  $C^{**}$  is the level of carryover that would be reached under the rule at the end of two "bumper-crop" years, that is, two successive years each with a yield of 35 bushels per acre, starting with an initial carryover of  $C^*$ .  $C^{**}$  also is given on a national aggregate basis including working stocks. The rules themselves are presented in terms of bushels per acre, rather than as a national aggregate, to make them directly applicable to situations where acreage planted differs.

All computations were carried out to the closest 0.01 bushel per acre. Some slight inaccuracy may be introduced in making the inversions by linear interpolation. The final results should be accurate to within 0.02 or 0.03 bushel per acre, and are almost certainly accurate to within 0.05 bushel per acre. These limits do not, of course, allow for errors in the estimates of the given conditions  $\gamma'$ ,  $\alpha$ ,  $\rho$ , and  $F$ .

TABLE 1.—*Corn, oats and barley, corn equivalent: Optimal carryover rules under specified conditions and related quantities*<sup>1</sup>

Item	Unit	Rule— $\hat{\theta}$											
		1	2	3	4	5	6	7	8	9	10	11	12
Condition:													
Elasticity— $\eta_0$ -----		-0.50	-0.50	-0.50	-0.50	-0.30	-0.30	-0.40	-0.50	-0.50	-0.50	-0.50	-0.50
Cost of storage— $\gamma'$ -----	Dol-----	.10	.10	.04	.04	.10	.04	.10	.10	.10	.04	.04	.10
Discount rate— $\alpha$ -----		.95	.98	.95	.98	.95	.98	.95	.95	.95	.98	.98	.95
Variability of yields— $\sigma$ -----	Bu-----	3.03	3.03	3.03	3.03	3.03	3.03	3.03	5.05	0	5.05	0	3.03
Optimal carryover per acre when supply per acre equals—													
28-----	do-----	0	0	0	0	0	0	0	0	0	0	0	0
29-----	do-----	0	0	0	0	0	.07	0	0	0	.33	0	0
30-----	do-----	0	0	0	.33	0	.77	0	0	0	1.03	0	0
31-----	do-----	0	.34	.46	.99	.55	1.50	.25	.28	0	1.75	.46	.33
32-----	do-----	.55	.93	1.07	1.69	1.19	2.25	.84	.90	.39	2.48	1.01	.87
33-----	do-----	1.13	1.57	1.74	2.41	1.86	3.02	1.47	1.53	.90	3.23	1.70	1.43
34-----	do-----	1.74	2.22	2.42	3.15	2.57	3.80	2.12	2.18	1.44	3.98	2.41	2.00
35-----	do-----	2.38	2.90	3.12	3.90	3.29	4.60	2.80	2.85	1.98	4.75	3.14	2.57
36-----	do-----	3.05	3.61	3.85	4.67	4.02	5.40	3.50	3.55	2.66	5.53	3.92	3.16
37-----	do-----	3.74	4.32	4.60	5.45	4.77	6.20	4.22	4.27	3.34	6.30	4.64	3.77
38-----	do-----	4.44	5.05	5.35	6.24	5.54	7.01	4.95	4.98	4.04	7.10	5.50	4.39
39-----	do-----	5.16	5.80	6.12	7.02	6.31	7.83	5.70	5.70	4.73	7.90	6.31	5.03
40-----	do-----	5.89	6.55	6.89	7.82	7.09	8.66	6.46	6.43	5.45	8.70	7.14	5.67
41-----	do-----	6.63	7.31	7.67	8.63	7.88	9.50	7.22	7.17	6.18	9.50	7.97	6.31
42-----	do-----	7.38	8.07	8.46	9.44	8.68	10.34	7.99	7.93	6.95	10.30	8.80	6.95
43-----	do-----	8.14	8.84	9.26	10.27	9.48	11.21	8.77	8.70	7.72	11.12	9.65	7.60
44-----	do-----	8.89	9.62	10.06	11.10	10.30	12.08	9.56	9.47	8.50	11.93	10.50	8.27
45-----	do-----	9.67	10.41	10.87	11.94	11.12	12.95	10.36	10.23	9.27	12.75	11.35	8.93
46-----	do-----	10.45	11.20	11.69	12.79	11.94	13.83	11.16	11.02	10.06	13.58	12.22	9.60

47	-----do-----	11.23	12.00	12.52	13.64	12.78	14.72	11.98	11.78	10.84	14.40	13.09	10.28
48	-----do-----	12.02	12.81	13.35	14.50	13.62	15.61	12.80	12.58	11.65	15.25	13.96	11.00
49	-----do-----	12.82	13.63	14.19	15.35	14.47	16.51	13.62	13.37	12.46	16.08	14.83	11.69
50	-----do-----	13.63	14.45	15.03	16.22	15.32	17.42	14.45	14.17	13.29	16.93	15.70	12.38
Related quantity:													
Per acre:													
k <sup>2</sup>	-----do-----	31.04	30.42	30.25	29.49	30.11	28.90	30.58	30.54	31.24	28.53	30.17	30.32
C*	-----do-----	.3	.5	.6	1.4	.7	2.7	.4	.4	0	3.0	0	.4
C**	-----do-----	4.1	5.3	5.7	7.8	6.1	10.1	5.0	5.0	-----	10.4	-----	4.3
As a National aggregate: <sup>3</sup>													
C*	Mil bu	242	270	284	396	298	578	256	256	200	620	200	256
C**	-----do-----	774	928	998	1,292	1,054	1,614	900	900	-----	1,656	-----	802

<sup>1</sup> The marginal value function is assumed to be linear for all rules except the last, where constant elasticity is assumed. See text for exact definition of symbols shown in stub.

<sup>2</sup> The value of S (supply per acre) below which the optimal carryover (exclusive of minimum working stocks) is zero.

<sup>3</sup> Obtained by multiplying the per acre value by 140 million acres, and adding 20 million bushels, the assumed minimum working stocks.

## FEED GRAINS:\* OPTIMAL CARRYOVER RULES PER ACRE

Under Alternative Conditions Specified in Table 1

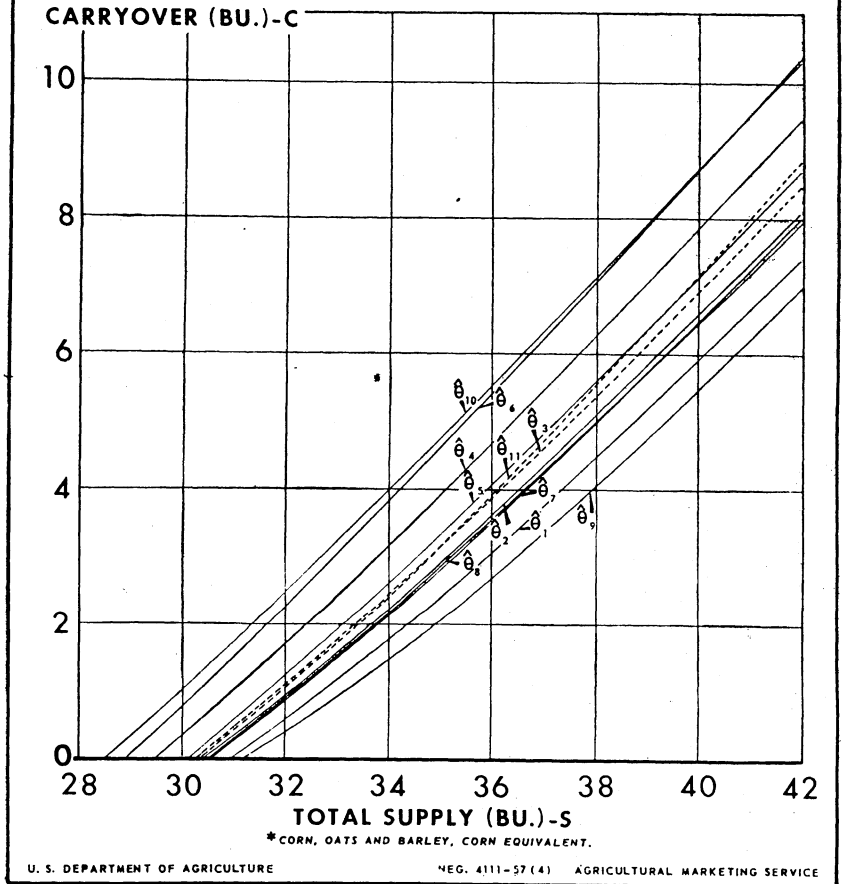


Figure 5.—Similarities, other than level, are greater than are differences even for the wide variety of conditions that apply to the 11 rules for which computations have been made, and for which the marginal value function is linear.

### SOME GENERAL CONCLUSIONS WITH RESPECT TO STORAGE

From the equations whose solution gives the optimal storage rule under conditions of stationarity (see page 46), we may derive the following conclusions (see pages 48-55):

1. If the marginal value function is the same as the market price function, the amounts which would be stored under an optimal governmental storage program are exactly the same as the amounts that would be stored in the aggregate by private firms in a so-called "idealized" free market. Such a market is one having

perfect competition, in the economic sense, and in which private firms seek to maximize their expected discounted profit.<sup>15</sup>

2. If the *marginal* cost of storage is a constant, so that the cost of storing each additional bushel of grain is the same regardless of whether large or small quantities are stored, and other conditions are the same as in (1) above, then the expected cost to the Government of operating an optimal storage program is zero. Under these circumstances, profits from the storage operations are just large enough to offset the costs of storage.

3. If the marginal value function is linear, the computations are somewhat simpler. However, as indicated by figure 4, the resulting storage rule is not linear, even in this case.

4. As illustrated in figure 3, the general shape (though not the position) of an optimal storage rule computed under the assumption of *no* variability in future yields is, at least in some cases, a fairly close approximation to the rule computed with the actual distribution of yields under the same conditions of total value, cost of storage, and interest rate.

Rules based on no variability in yields can be computed fairly easily (see pages 33 and 52). Moreover, it can be shown that the initial assumption in the iterative process can be any arbitrary storage rule and the iterations still will converge to the optimal rule. Furthermore, the closer the initial assumption is to the ultimate optimal rule, the fewer are the iterations required. These facts permit us to use rules that have been computed under the assumption of zero variability in yields for two purposes: (a) To reduce the number of iterations required for the process that leads to the actual optimal rule with yield variability included in the solution by providing reasonably accurate first approximations as a starting point, and (b) to provide rough but easily obtained measures of the effects on the optimal rule of making changes in the estimates of the other conditions.

5. When the quantity to be stored is plotted on the vertical scale and the total supply is plotted on the horizontal scale (as in figures 1 through 5), the optimal quantity to be stored increases continually with increasing supply except that, when supplies are smaller than some specified amount, the quantity to be stored (in excess of minimum working stocks) is zero. The approximate point at which the curve cuts the supply axis can be determined rather easily (see pages 36, 54).

6. Use of (4) and (5) above gives a convenient method for obtaining a first approximation to the optimal rule under a specified set of circumstances. The following three steps are involved: (a) Compute the rule with zero variability in yields; (b) compute the approximate point at which the curve cuts the supply axis when yields vary in their normal way; and (c) shift the curve based on no variability in yields horizontally to the left on the graph so that it cuts the supply axis at the indicated point. This gives an approximation to the rule when yields vary in their normal way. Use of this approach is described in detail in the next section.

## A METHOD FOR OBTAINING APPROXIMATIONS TO THE RULES

To illustrate the procedure, approximations to optimal rules under two sets of conditions are obtained. The conditions are those used for rules  $\theta_1$  and  $\theta_6$  as shown in table 1. As noted in the preceding paragraph, the first step is to compute rules using the specified conditions but based on an assumed yield variability in future years of zero. To avoid possible confusion with the accurately-calculated rules given in table 1, we refer to these rules as A and B, respectively, and label the corresponding rules obtained when yields are assumed constant as A' and B', respectively.

To facilitate the computations, we show in table 2 the conditions that relate to these rules. Items in the first three rows are the same as the comparable items in table 1; those in the next two rows were obtained by the method discussed on page 24 (see equations (9) and

<sup>15</sup> For a more precise statement of the conditions under which this conclusion is valid, see pages 48-49.

(11)). The last item is obtained by making use of the constant term and the slope coefficient for the marginal value function in connection with utilization of 29.46 bushels per acre, the assumed average yield. For rule A, this computation is made in the following way:

$$\$4.50 - (0.10 \times 29.46) = \$1.554$$

TABLE 2.—Conditions used in obtaining optimal rules A and B

Item	Rules	
	A	B
Marginal cost of storage, $\gamma'$ , dollars per bushel.....	0. 10	0. 04
Discount factor, $\alpha$ , $1/(1+\text{interest rate})$ .....	. 95	. 98
Elasticity when utilization is 30 bushels per acre, $\eta_0$ .....	-. 50	-. 30
Marginal value function: $\rho(Y) = a - bY$		
Constant term, $a$ .....	4. 50	6. 50
Slope coefficient, $-b$ .....	-. 10	-. 167
Marginal value when utilization equals 29.46 bushels, $\rho_0$ , dollars.....	1. 554	1. 590

*Estimating rules when yields are assumed constant.*—The computations for rules A' and B', respectively, are shown in table 3. Numbered items in the remainder of this paragraph relate to the columns of that table. (1) The number of the row. The symbol  $i$  is used to indicate the row in subsequent columns. Note that  $i=0$  for the first row. (2) The discount factor raised to the  $(i+1)$  power. (3) The sum from  $j=0$  to  $j=i$  of the discount factor raised to the  $j^{\text{th}}$  power. Any number raised to the  $0^{\text{th}}$  power equals 1. Hence, for  $i$  equal zero, the number in this column is 1. The item in the second row equals  $1 + (0.95)^1 = 1.95$ . The item in the third row equals  $1 + (0.95)^1 + (0.95)^2 = 2.852$ . The series can be conveniently obtained by inserting a 1 in the first row and adding the item from the  $i-1$  row of the second column to the cumulative total to obtain the item in the  $i^{\text{th}}$  row of the third column. (4) Column (2) times the marginal value when utilization equals 29.46 bushels per acre. (5) The marginal storage cost times column (3). (6) Column (4) minus column (5). (7) The reciprocal of the absolute value of the slope coefficient for marginal value  $(1/b)$  times column (6). (8) The constant in the equation for marginal value divided by the absolute value of the slope coefficient  $(a/b)$  minus column (7). For rule A', the quotient is obtained as follows:  $4.50/0.10 = 45.00$ . (9) Total supply equals carryover plus column (8). Each entry in this column is obtained after the corresponding entry in column (10) has been computed. Column (10). Carryover equals supply in the preceding row (column 9) minus the assumed average yield of 29.46 bushels per acre.

In carrying out these computations, we first fill in all values in column (1), all items in column (2), and so forth through column (8). Items in column (2) can be obtained by successively multiplying the item in the preceding row by the discount factor. We have already described a convenient method for obtaining the items in column (3). Items in columns (4), (5), and (7) are obtained by multiplying the items in a previously computed column by a constant. Items in

columns (6) and (8) are obtained by subtraction. The item in the first row of column (10) always is zero. Given this value, we can obtain a value for the item in the first row of column (9). From this, we obtain the item in the second row of column (10). This permits us to obtain a value for the item in the second row of column (9). By repeating this process, all items in columns (10) and (9) are obtained. The amount of clerical work involved is not great. Successive iterations are continued until a sufficient range in observations for supplies and carryovers are obtained. Thus, 5 points are computed in table 3 for rule A' and 10 for rule B'.<sup>16</sup>

TABLE 3.—*Corn, oats, and barley: Computations involved in obtaining rules A' and B'*<sup>1</sup>

Rule A'									
(1) Row (i)	(2) $\alpha^{(i+1)}$	(3) $\sum_{j=0}^i \alpha^j$	(4) $(2) \times p_0$	(5) $(3) \times \gamma'$	(6) $(4) - (5)$	(7) $(6) \times 1/b$	(8) $a/b - (7)$	(9) $(8) + (10)_s$	(10) $(9)_{i-1} - 29.46^2$ $c = \theta(s)$
0 ----	0.950	1.000	1.477	0.100	1.377	13.77	31.23	31.23	0
1 ----	.902	1.950	1.402	.195	1.207	12.07	32.93	34.70	1.77
2 ----	.857	2.852	1.332	.285	1.047	10.47	34.53	39.77	5.24
3 ----	.814	3.709	1.264	.371	.893	8.93	36.07	46.38	10.31
4 ----	.773	4.523	1.201	.452	.749	7.49	37.51	54.43	16.92

Rule B'									
0 ----	.980	1.000	1.559	.040	1.519	9.11	29.89	29.89	0
1 ----	.961	1.980	1.526	.079	1.447	8.68	30.32	30.75	.43
2 ----	.941	2.941	1.494	.118	1.376	8.25	30.75	32.04	1.29
3 ----	.922	3.882	1.465	.155	1.310	7.86	31.14	33.72	2.58
4 ----	.904	4.804	1.437	.192	1.245	7.47	31.53	35.79	4.26
5 ----	.886	5.708	1.408	.228	1.180	7.08	31.92	38.25	6.33
6 ----	.868	6.594	1.379	.264	1.115	6.68	32.32	41.11	8.79
7 ----	.851	7.462	1.352	.298	1.054	6.33	32.67	44.32	11.65
8 ----	.834	8.313	1.325	.333	.992	5.95	33.05	47.91	14.86
9 ----	.817	9.147	1.298	.366	.932	5.59	33.41	51.86	18.45

<sup>1</sup> See text for computations involved in each column.

<sup>2</sup> Based on values in the preceding row. A zero always is used in this column for row 0.

Results from these computations are shown in figure 6, along with those indicated when yields are assumed to vary in a normal way, based on data for  $\hat{\theta}_1$  and  $\hat{\theta}_6$  from table 1. A' is roughly parallel to A, and B' to B, indicating that a drastic change in one's assumption about the variability of yields does not change the slope of the optimal storage rule very much, although the position of the curve does change. The reader also should note that the difference between A and A' or between B and B' is much less than the difference between either A and B or between A' and B', confirming the view that assumptions

<sup>16</sup> For the mathematics underlying these computations, see pages 52-54 and Appendix note 9.

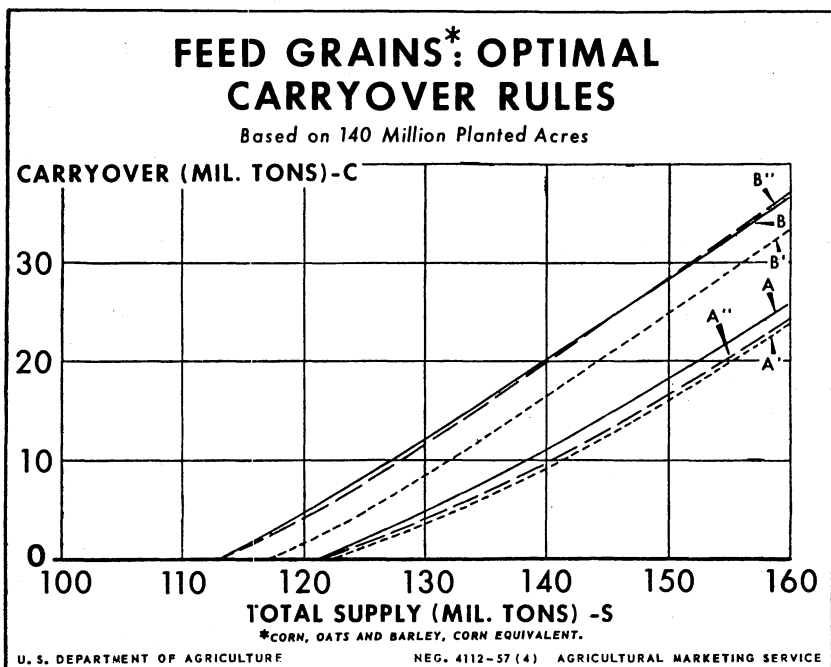


Figure 6.—An approximate method for obtaining optimal storage rules which requires only simple arithmetic operations gives results that are nearly identical to those obtained by the complete mathematical technique. The true rules are labeled A and B and the corresponding approximate rules, A'' and B''. A' and B' were obtained as an intermediate step.

about the variability in yields have less effect on the optimal rules than do changes in assumptions about other conditions.

*Estimating the supply below which no grain should be stored when yields vary.*—This section describes a relatively simple method that can be used to approximate the supply below which no grain (in excess of minimum working stocks) should be stored when yields vary in their normal way. This point is referred to here by the symbol  $k$ . To obtain this approximation to  $k$ , it is first necessary to calculate a function  $L$  and a constant  $M$ .  $L$  is a function of  $k$  itself, the form of the function depending on the probability distribution of outputs.  $M$  is a constant, calculated from the marginal value function, average output, the marginal cost of storage, and the discount factor.

We first obtain a table showing the values of  $L$  that are associated with specified values of  $k$ . The values of  $k$  are taken at the lower class limits of the intervals for yield per planted acre shown in the tabulation on page 22. Values of  $L$  are obtained by making use of the data shown in the tabulation. They can best be obtained by starting with the largest value of  $k$  and working backward. The computations are shown in table 4. Numbered items in the remainder of this paragraph relate to the columns in that table. (1) Lower limit of the class intervals for yields shown in the tabulation on

page 2 starting with the last or largest yield. This is the value of  $k$ . (2) Cumulative frequency of the yields, starting with the largest yield (see page 22). (3) Column (1) times column (2). (4) Midpoint of the class interval for yields (see page 22). (5) Frequency for that yield (see page 22). (6) Cumulative product of the items in column (4) times those in column (5). (7) Column (6) minus column (3). This is the value of  $L$ . The reader will note that computations for some values of  $k$  are omitted. Computations involved for these rows are clear from those shown in the table.

TABLE 4.—*Corn, oats, and barley: Computations involved in obtaining  $L$  for specified values of  $k$ <sup>1</sup>*

(1) $k$	(2) $\Sigma f(x)$	(3) (1) $\times$ (2)	(4) $x$	(5) $f(x)$	(6) $\Sigma (4) \times (5)$	(7) $L$ (6) - (3)
34.5-----	0.02	0.69	35	0.02	0.70	0.01
33.5-----	.02	.67	34	.00	.70	.03
32.5-----	.12	3.90	33	.10	4.00	.10
31.5-----	.22	6.93	32	.10	7.20	.27
30.5-----	.36	10.98	31	.14	11.54	.56
29.5-----	.56	16.52	30	.20	17.54	1.02
19.5-----	.98	19.11	20	.02	29.08	9.97
18.5-----	1.00	18.50	19	.02	29.46	10.96

<sup>1</sup> See text for computations involved in each column.

The next step is to obtain a value for  $M$ . This is done by use of the following formula. The constant term and the regression coefficient referred to are for the marginal value function shown in table 2. The absolute (or positive) value of the regression coefficient is used. The discount factor and the marginal cost of storage are given in table 2, and the average production is the mean of the distribution of yields shown on page 22.

$$M = (\text{Discount factor}) (\text{average production}) + \frac{(1 - \text{discount factor}) (\text{constant term})}{|\text{regression coefficient}|} + \frac{(\text{marginal cost of storage})}{|\text{regression coefficient}|} \quad (12)$$

For rule A, use of this formula gives the following:

$$M = (0.95)(29.46) + \frac{(1.00 - 0.95)(4.50)}{(0.10)} + \frac{(0.10)}{(0.10)} = 31.24$$

For rule B,  $M = 29.88$ .

By making use of  $M$ , the discount factor, and an estimate of the slope of the optimal rule we now obtain a second set of values that shows a relationship between  $k$  and  $L$ . The value of  $\hat{k}$  that we desire

is the value of  $k$  that satisfies each of these relations. The second relation between  $k$  and  $L$  is obtained from the formula:

$$k = M - (\text{discount factor})(\text{slope})L \quad (13)$$

In the computations shown in the next two paragraphs, we use an approximate value of 0.6 for the slope of the storage rule.

For rule A, formula (13) gives the following:

$$\begin{aligned} k &= 31.24 - (0.95)(0.6)L \\ &= 31.24 - 0.57L \end{aligned}$$

By comparing this formula with the values of  $k$  and  $L$  shown in table 4, we see that the value of  $k$  that will satisfy both equations lies between 31.5 and 30.5 bushels. For these values of  $k$ ,  $L$  in table 4 has a value of 0.27 and 0.56, respectively. We use these values of  $L$  in the formula shown above and solve for  $k$ . Results obtained are 31.09 and 30.92 bushels, respectively. We now make a greatly enlarged graph with  $k$  on the vertical scale and  $L$  on the horizontal scale. Values for  $k$  of 31.5 and 31.09 are plotted opposite a value for  $L$  of 0.27; values for  $k$  of 30.5 and 30.92 are plotted opposite a value for  $L$  of 0.56. The points for which  $k$  equals 31.5 and 30.5 are connected with a line, as are the points for which  $k$  equals 31.09 and 30.92. The value for  $k$  at the intersection of the two lines is the desired  $\hat{k}$ . For rule A, this is 31.01 bushels and for rule B, 29.14 bushels.

A modification can be made in estimating  $k$ , namely that of using for the slope of the optimal rule the slope obtained under similar conditions when yields are assumed to be constant. As the function is a curve, a decision has to be reached as to the point on the curve for which the slope is to be computed. Two points appear relevant: (1) The point closest to  $\hat{k}$ , and (2) the point closest to the average supply. The average supply can be obtained by adding to the constant production the average carryover. For rule A this is  $29.46 + 0.3 = 29.76$ . As this point is below  $\hat{k}$ , we can compute the slope for only one segment of the curve, namely that shown by the first 2 rows of table 3. The average slope for this segment equals  $(1.77 - 0) / (34.70 - 31.23) = 0.51$ . When this slope is used in formula (13), a value for  $\hat{k}$  of 31.05 bushels is given, almost the same as the true value of 31.04 (see table 1).

For rule B, the average supply equals  $29.46 + 2.7 = 32.16$ . This is somewhat above  $\hat{k}$ . Hence, two sets of computations were made, one for which the slope was estimated over the segment of the curve shown in the first 2 rows of the second section of table 3 and the other over the second, third, and fourth rows, as the average supply is just about at the midpoint of this segment. The average slope in the first case was 0.50, and in the second case, 0.67. When these values were used in the formula, estimates for  $k$  of 29.33 and 28.86 bushels, respectively, were given, compared with the true value of 28.90.

Results of estimating  $\hat{k}$  when the estimates are based on the several different ways of estimating the slope of the optimum rule are tabu-

lated in table 5. The results obtained suggest that an efficient way to obtain this value is to use as an estimate for the slope the slope for the rule when yields are assumed not to vary measured at a point close to the average supply. By using this approach and the general method for obtaining the rule when yields are assumed not to vary described beginning on page 34, a close approximation to the rule can be obtained. Mathematical techniques required to develop these computational methods and to show just what the various steps mean are described on pages 54-55 and in Appendix Note 10.

TABLE 5.—Corn, oats, and barley: Estimates of  $\hat{k}$  by specified methods, actual and as a difference from the true value

Method of estimating slope of optimal rule	Estimate of $\hat{k}$			
	Actual value for rule—		Difference from true value for rule—	
	A	B	A	B
Arbitrary value of 0.6.....	<i>Bushels</i> 31. 01	<i>Bushels</i> 29. 14	<i>Bushels</i> —0. 03	<i>Bushels</i> 0. 24
Same slope as for rule when yields are assumed not to vary measured at the point closest to— $\hat{k}$ .....	31. 05	29. 33	. 01	. 43
Average supply.....	31. 05	28. 86	. 01	— . 04

*Final results when yields vary.*—The actual rules obtained by the several methods are shown in figure 6. The notation is the same as that given in the text, but it may be helpful to the reader to review briefly the methods used in obtaining them. Rules labeled A and B were obtained by use of the full mathematical procedure based on the yield distribution shown in the tabulation on page 22 and the other variables shown in table 2. Data that relate to these rules in terms of bushels per acre are shown in table 1 under rules  $\hat{\theta}_1$  and  $\hat{\theta}_2$ , respectively. For the chart, these were converted to million tons, assuming that 140 million acres need to be planted to the 3 feed grains to meet utilization requirements when yields are at their average level. The supply range shown on the chart goes to a maximum somewhat above the maximum supply of the four feed grains on record when research on this study was completed.<sup>17</sup>

Rules A' and B' are based on assumptions similar to those used for rules A and B, respectively, except that yields in future years are assumed to be constant at their average level. Computations involved in obtaining these rules, in terms of bushels per planted acre, are shown in table 3. Rules A'' and B'' were obtained by using the

<sup>17</sup> Namely, a supply of 153 million tons and a carryover of 31 million tons for the marketing year beginning October 1950. New records were set for the years beginning in 1954, 1955, and 1956. In the last named year, supply was 174 million tons and the carryover, 43 million tons.

values computed for rules A' and B', respectively, adjusted in such a way that the curves pass through the point on the supply axis equal to the value of  $\hat{k}$  shown in the last row of table 5. Methods by which the values in table 5 were obtained are discussed on pages 36-39. As for the other rules, conversions from bushels per acre to million tons also were made.

It is evident that, particularly for the assumptions used in connection with rule B, the approximations obtained by the last approach are nearly identical with the rules obtained by applying the complete mathematical technique.

## MATHEMATICAL SOLUTIONS

Some of the symbols used in the mathematical solution have been introduced in earlier sections. As they were not necessarily defined in strict mathematical terms, it now appears desirable to repeat the definitions, making use of additional rigor where required.

As indicated on page 11, the conditions which are relevant and which must be estimated prior to the derivation of storage rules are the following: (1) The discount factor  $\alpha$  (equal to  $1/(1+r)$ , where  $r$  is the interest rate), (2) the cost of storage, (3) the conditions of utilization (demand), and (4) the conditions of production (supply) of the grain. The latter three sets of conditions are conveniently handled by setting up the following functions:

$\gamma_t(C)$ : the cost (in dollars) of carrying over the quantity  $C$  in the year  $t$ ;

$\delta_t(Y)$ : the total value to the general public (measured in dollars) attributable to the utilization of the quantity  $Y$  in year  $t$ ; and

$F_t(x)$ : the probability distribution of output  $x$  in year  $t$ .

As an alternative to the use of the total value function  $\delta$ , we may, if  $\delta$  is differentiable, use the marginal value function  $\rho_t(Y)$ , defined as the derivative of  $\delta_t(Y)$ . For many purposes, use of the marginal value function  $\rho$  turns out to be more convenient and more illuminating than use of the total value function  $\delta$ . However, our initial presentation of the solution is in terms of the  $\delta$  function, since the exposition and proofs are more straightforward in those terms.

Problems relating to the determination and estimation of these functional relationships were discussed in earlier sections. To specify the criterion of optimality, first, "the gain" incurred in year  $t$ ,  $W_t$ , is defined as the total value of the grain utilized minus the cost of storage, that is:

$$W_t = \delta_t(Y_t) - \gamma_t(C_t) \quad (14)$$

$$= \delta_t(S_t - C_t) - \gamma_t(C_t) \quad (15)$$

since the quantity utilized,  $Y_t$ , equals the total supply,  $S_t$ , minus the amount carried over,  $C_t$ .

As indicated on page 9, a "rule of storage" is a function ( $\theta_t$ ) which explicitly states the dependence of  $C_t$  on  $C_{t-1}$  and  $X_t$ , that is:

$$C_t = \theta_t(C_{t-1}, X_t) \quad (4)$$

A "storage policy" for a period of  $n$  years ( $t=1, \dots, n$ , where the current year is designated as 1) is defined as a set of storage rules  $(\theta_1, \dots, \theta_n)$ .

If we consistently follow a set of storage rules, the total gain in any year depends on the initial supply and our storage rule. In thinking about a year in the future, say year  $t$ ,  $X_t$  is unknown, so we make use of its probability distribution  $F_t(x_t)$ . By using these probability distributions we can, for a given set of storage rules, obtain an expected value for  $W_t$ , namely  $EW_t$ . That is, given the probability distributions of output  $F_2, \dots, F_t$ , if  $\theta_1, \dots, \theta_t$  are known, we could conceptually, if not practically, find  $EW_t$  by a  $(t-1)$ -tuple integration over  $F_2, \dots, F_t$ . Now let  $V_{1,n}$  be the sum of expected gains in years 1,  $\dots$ ,  $n$  discounted back to the year 1. If the annual discount factor is a constant  $\alpha$  ( $0 < \alpha < 1$ ), then the discount factor applicable in year 1 to values occurring in year  $t$  is  $\alpha^{t-1}$ , so that

$$V_{1,n} = W_1 + \alpha EW_2 + \alpha^2 EW_3 + \dots + \alpha^{n-1} EW_n \quad (16)$$

For given  $F_2, \dots, F_n$ ,  $V_{1,n}$  is a function of  $S_1$  and  $\theta_1, \dots, \theta_n$ , since  $EW_t$  is a function of  $S_1$  and  $\theta_1, \dots, \theta_t$ .

We now define the optimal storage policy as that set of rules  $\hat{\theta}_1, \dots, \hat{\theta}_n$  which maximizes  $V_{1,n}$  for any  $S_1$ .

*The solution based on the total value function.*—First rewrite equation (15), to simplify the notation, as:

$$W_t = W_t(S_t, C_t) \quad (15.1)$$

That is, the gain in any year is a function of total supply and carry-over.

For every possible value of  $S_n$  in the  $n$ th year, find  $C_n$  to maximize  $W_n(S_n, C_n)$ . This gives  $C_n$  as a function of  $S_n$ , and that function is  $\hat{\theta}_n$ , the optimal storage rule for the  $n$ th year.<sup>18</sup> With  $\hat{\theta}_n$  thus determined, the maximized gain in the  $n$ th year is a function of  $S_n$  alone and may be designated  $\hat{V}_{n,n}(S_n)$ .

Proceed back to the year  $n-1$ . From equation (1),

$$S_n = C_{n-1} + X_n \quad (1.1)$$

where  $X_n$  (from the viewpoint of year  $n-1$ ) is a random variable with probability distribution  $F_n(x_n)$ . To get the expected value (in year

<sup>18</sup> With the cost of storage function and the total value function both monotonically increasing,  $C_n$  always equals zero. This result is not necessary, however, to what follows. In particular, if it is decided that for some reason stocks should be at some specified level, say  $C_n$ , at the end of a specified  $n$ -year period, this value of  $C_n$  may simply be inserted into the solution, and the procedure outlined then leads to the maximum-gain or least-loss program for bringing stocks to that level. For the case of stationarity, where the criterion is to maximize the sum of discounted expected gains in all future years, the situation is different:  $C_n$  may be set at any arbitrary value (zero is usually as convenient as any), and the solution, that is, the optimal stationary storage rule, is independent of, or completely unaffected by, the value so set; in this case, the value of  $n$  is also unspecified: the computational iterations are simply continued until convergence is achieved.

$n-1$ ) of the gain in year  $n$  (as maximized by  $\hat{\theta}_n$ ), we integrate  $\hat{V}_{n,n}(C_{n-1}+x_n)$  over the probability distribution  $F_n(x_n)$ , leaving a function of  $C_{n-1}$  alone. That is,

$$E\hat{V}_{n,n}(C_{n-1}+x_n)=Q_{n-1}(C_{n-1}) \quad (17)$$

This expression represents the expected (maximized) gain in year  $n$  as a function of carryover in year  $n-1$ . In year  $n-1$ , then, for every possible value of  $S_{n-1}$ , we find the value of  $C_{n-1}$  to maximize the gain in year  $n-1$  plus the discounted expected gain in year  $n$ . That is, we maximize

$$V_{n-1,n}(S_{n-1}, C_{n-1})=W_{n-1}(S_{n-1}, C_{n-1})+\alpha Q_{n-1}(C_{n-1}) \quad (18)$$

This gives  $C_{n-1}$  as a function of  $S_{n-1}$ , and that function is  $\hat{\theta}_{n-1}$ , the optimal storage rule for year  $n-1$ . With  $\hat{\theta}_{n-1}$  thus determined, the sum of maximized expected gains in years  $(n-1, n)$  (discounted to year  $n-1$ ) is a function of  $S_{n-1}$  alone, and may be designated  $\hat{V}_{n-1,n}(S_{n-1})$ .

Proceed back to year  $n-2$ . From equation (1),

$$S_{n-1}=C_{n-2}+X_{n-1} \quad (1.2)$$

To get the expected value (in year  $n-2$ ) of the sum of the gains in years  $(n-1, n)$  (discounted to year  $n-1$ , and maximized by  $\hat{\theta}_{n-1}$ ,  $\hat{\theta}_n$ ), we integrate  $\hat{V}_{n-1,n}(C_{n-2}+x_{n-1})$  over the probability distribution  $F_{n-1}(x_{n-1})$ , giving a function of  $C_{n-2}$  alone, say  $Q_{n-2}(C_{n-2})$ . In year  $n-2$ , then, for every possible value of  $S_{n-2}$ , we find the value of  $C_{n-2}$  to maximize the gain in year  $n-2$  plus the discounted expected sum of gains in years  $(n-1, n)$ . That is, we maximize

$$V_{n-2,n}(S_{n-2}, C_{n-2})=W_{n-2}(S_{n-2}, C_{n-2})+\alpha Q_{n-2}(C_{n-2}) \quad (18.1)$$

This gives  $C_{n-2}$  as a function of  $S_{n-2}$ , and that function is  $\hat{\theta}_{n-2}$ , the optimal storage rule for year  $n-2$ . With  $\hat{\theta}_{n-2}$  thus determined, the sum of maximized expected gains in years  $(n-2, n-1, n)$  (discounted to year  $n-2$ ) is a function of  $S_{n-2}$  alone, and may be designated  $\hat{V}_{n-2,n}(S_{n-2})$ .

The general procedure now can be seen for determining the optimal storage rule in year  $t$ ,  $\hat{\theta}_t(1 \leq t < n)$ , once the optimal rules in succeeding years  $\hat{\theta}_{t+1}, \dots, \hat{\theta}_n$  are determined. The sum of expected gains in years  $(t+1, \dots, n)$  (maximized by  $\hat{\theta}_{t+1}, \dots, \hat{\theta}_n$ , and discounted to year  $t+1$ ) is a known function of  $S_{t+1}$ , say  $\hat{V}_{t+1,n}(S_{t+1})$ . Since by equation (1)

$$S_{t+1}=C_t+X_{t+1} \quad (1.3)$$

where (from the viewpoint of year  $t$ )  $X_{t+1}$  is a random variable with the probability distribution  $F_{t+1}(x_{t+1})$ , the expected value (in year  $t$ )

of  $\hat{V}_{t+1,n}$  is obtained by integrating  $\hat{V}_{t+1,n}(C_t + x_{t+1})$  over that probability distribution, giving a function of  $C_t$  alone, say  $Q_t(C_t)$ . Then, for every possible value of  $S_t$ , we find the value of  $C_t$  which maximizes the gain in year  $t$  plus the discounted expected sum of (discounted, maximized) expected gains in years  $(t+1, \dots, n)$ . That is, we maximize

$$V_{t,n}(S_t, C_t) = W_t(S_t, C_t) + \alpha Q_t(C_t) \quad (18.2)$$

This gives  $C_t$  as a function of  $S_t$ , and that function is  $\hat{\theta}_t$ , the optimal storage rule in year  $t$ . It also gives the maximized sum of expected gains in years  $(t, \dots, n)$  (discounted to year  $t$ ) as a function of  $S_t$ , say  $\hat{V}_{t,n}(S_t)$ .

Continuing back to year 1 (the current year), we have thus determined the optimal storage rule for each year  $\hat{\theta}_1, \dots, \hat{\theta}_n$ , and also the maximized sum of discounted expected gains in all the years, as a function of  $S_1$ ,  $\hat{V}_{1,n}(S_1)$ .<sup>19</sup> The computational operations must be carried out numerically, that is, by using discrete values of the various functions corresponding to selected discrete values of their respective arguments. The essential reason for this is the existence of discontinuities caused by the restriction of  $C$  (carryover) to non-negative values. The author, at least, has been unable to find any analytical "tricks," even under the most simplifying assumptions about the forms of the relevant functions, which make possible a non-numerical computational procedure.

We next consider modifications in the procedure when the conditions are assumed to be stationary. By "stationarity" is meant the condition that the annual discount factor  $\alpha$  and the three functions, cost of storage  $\gamma$ , social value  $\delta$ , and distribution of output  $F$ , are the same in every year; by "independence" is meant the condition that each of the three functions is unaffected by any variable other than its explicitly stated argument ( $C$ ,  $Y$  or  $x$  respectively). The solution outlined above assumes independence in the functions, but not stationarity. If the independence condition does not hold, the general form of the solution is the same as that outlined, but it becomes a bit more complicated, and the computational requirements may become much greater. If stationarity is assumed, as well as independence, and if the desired storage policy is that set of rules which will maximize, in each year, the sum of discounted expected gains in all future years, then the resulting optimal storage rules are identical for all years. That is, a single (stationary) optimal rule  $\hat{\theta}$  applies in every year.

<sup>19</sup> For a more concise, more purely symbolic statement of the procedure, which may help to clarify both its nature and the specific steps, see Appendix Note 5. A characteristic of the method is that the computational operations are performed on the successive gain functions themselves, the resulting storage rules falling out more or less incidentally.

It should be noted that functions like  $\hat{V}_{t+1,n}(C_t + x_{t+1})$  are not, in general, linear, so that, to get the expected value, it is necessary to integrate the whole function over the distribution of  $x$ , rather than simply to insert the expected value of  $x$ . That is, in general,  $E\hat{V}_{t+1,n}(C_t + x_{t+1})$  is *not* equal to  $\hat{V}_{t+1,n}(C_t + Ex_{t+1})$ .

Such a rule can be computed by the iterative procedure just outlined, by taking the number of iterations successively larger. The procedure can be summarized as follows: Define an operator  $J$  operating on any function  $\phi$  by

$$J\phi(S) = \max_{0 \leq C \leq S} [\delta(S-C) - \gamma(C) + \alpha E\phi(C+x)] \quad (19)$$

where  $E$  means the mathematical expected value with respect to  $x$ , that is, the integral over the probability distribution  $F(x)$ . Then

$$\hat{V}_{1,n}(S) = J^{n-1} \delta(S) \quad (20)$$

where  $\hat{V}_{1,n}$  is the maximized sum of discounted expected gains in years 1 through  $n$ ,  $S$  is the initial year's total supply (the subscript 1 is omitted for convenience), and the superscript  $n-1$  on  $J$  indicates that the operation is performed  $n-1$  successive times. As  $n$  increases,  $\hat{V}_{1,n}(S)$  converges to a limit, that is,

$$\lim_{n \rightarrow \infty} J^n \delta(S) = \beta(S) \quad (21)$$

where  $\beta(S)$  is the maximized sum of discounted expected gains in all future years. This sum is a function of the initial year's total supply,  $S$ .<sup>20</sup>

Another way to look at the problem is to say that we want to find the function  $\beta(S)$  which satisfies the equation

$$J \beta(S) = \beta(S) \quad (22)$$

The uniqueness of the solution, shown in the convergence proof, depends on  $\delta$  being bounded (within the range of possible values of  $S$ ) and on the annual discount factor  $\alpha$  being less than 1. Having obtained  $\beta(S)$ , the optimal stationary storage rule  $\hat{\delta}$  is obtained by observing, for each value of  $S$ , that value of  $C$  which maximizes

$$\delta(S-C) - \gamma(C) + \alpha E \beta(C+x) \quad (19.1)$$

*The solution based on the marginal value function.*—Although the method suggested on page 13 for determining the total value function implies that it is differentiable, the method of solution outlined above does not require either differentiability or continuity in this function. However, if the total value function  $\delta(Y)$  is differen-

<sup>20</sup> A detailed discussion, and the proof of convergence, is given in Appendix Note 5.

In computations for practical applications, the iterations are not, of course, continued to infinity, but only to the point where convergence is achieved. That is, to the point where  $J^n \delta(S) = J^{n-1} \delta(S)$ , for all relevant values of  $S$ . Once such convergence is obtained, further iterations in no way change the results. The number of iterations required depends on the conditions of the particular application, and also on the accuracy of the basic data or the number of significant digits carried in the computations. The larger the number of significant digits carried, the larger the number of iterations required to produce complete convergence.

tiable, then optimal storage rules  $\hat{\theta}_1, \dots, \hat{\theta}_n$ , or, under stationarity, the optimal storage rule  $\hat{\theta}$ , can be obtained using the marginal value function  $\rho(Y)$ , defined as the derivative of total social value, that is,

$$\rho(Y) = \frac{d\delta(Y)}{dY} \quad (23)$$

Each of the successive steps in the solution can be shown to be mathematically equivalent to the corresponding step in the procedure using the total value function  $\delta(Y)$ .

As before, we start with the last year of the  $n$ -year period. The following steps are involved:

1. For year  $n$ , set the carryover equal to  $C_n$ . If the policy criterion is to maximize the sum of discounted expected gains over the  $n$ -year period, then  $C_n = 0$ . If the criterion is to have a specified level of stocks on hand at the end of the period, and to maximize the sum of discounted expected gains during the period subject to that restraint, then set  $C_n$  equal to that specified level.

2. For year  $n-1$ , find for each possible value of  $S$  the value of  $C > 0$  which satisfies

$$\alpha E \rho_n(C + x_n - C_n) - \gamma'_{n-1}(C) - \rho_{n-1}(S - C) = 0 \quad (24)$$

where  $E$  is the mathematical expectation with respect to  $x_n$  (the integral over the probability distribution  $F_n(x)$ ),  $\alpha$  is the annual discount factor,  $\rho_n$  is the marginal value function in year  $n$ ,  $\gamma'_{n-1}(C)$  is the marginal cost of storage in year  $n-1$  (the derivative of  $\gamma_{n-1}(C)$ ), and  $\rho_{n-1}$  is the marginal value function in year  $n-1$ . This gives  $C$  as a function of  $S$ , and that function is the optimal storage rule in year  $n-1$ ,  $\theta_{n-1}(S)$ . For those values of  $S$  where no value of  $C > 0$  satisfies the above condition,  $\theta_{n-1}(S) = 0$ .

3. For year  $n-2$ , find for each possible value of  $S$  the value of  $C > 0$  which satisfies

$$\alpha E \rho_{n-1}[C + x_{n-1} - \hat{\theta}_{n-1}(C + x_{n-1})] - \gamma'_{n-2}(C) - \rho_{n-2}(S - C) = 0 \quad (24.1)$$

where  $E$  is the mathematical expectation with respect to the random variable  $x_{n-1}$ ,  $\hat{\theta}_{n-1}$  is the optimal storage rule for year  $n-1$  (determined in the preceding step), and the other symbols are similar to those used in equation (24). This gives  $C$  as a function of  $S$ , and that function is the optimal storage rule in year  $n-2$ ,  $\hat{\theta}_{n-2}(S)$ . For values of  $S$  where no value of  $C > 0$  satisfies the condition,  $\hat{\theta}_{n-2}(S) = 0$ .

4. In general, for year  $t$  ( $t = n-1, n-2, \dots, 1$ ), once the optimal storage rule for year  $t+1$ ,  $\hat{\theta}_{t+1}$ , is determined, find for each possible value of  $S$  the value of  $C > 0$  which satisfies

$$\alpha E \rho_{t+1}[C + x_{t+1} - \hat{\theta}_{t+1}(C + x_{t+1})] - \gamma'_t(C) - \rho_t(S - C) = 0 \quad (24.2)$$

where  $E$  is the mathematical expectation with respect to the random variable  $x_{t+1}$ , and the other symbols are similar to those defined in preceding steps. This gives  $C$  as a function of  $S$ , and that function is the optimal storage rule for year  $t$ ,  $\hat{\theta}_t(S)$ . For values of  $S$  where no value of  $C > 0$  satisfies the condition,  $\hat{\theta}_t(S) = 0$ .

The optimal storage rule for each year of the  $n$ -year period,  $\hat{\theta}_1, \dots, \hat{\theta}_n$ , is thus determined.<sup>21</sup>

<sup>21</sup> Proof of the mathematical equivalence of this procedure to that using the total value function is given in Appendix Note 6.

It should be noted that functions like

$$\rho_t[C + x_t - \theta_t(C + x_t)]$$

are not, in general, linear in  $x_t$ , even if  $\rho_t$  is linear; so that, to get the expected

In almost any conceivable practical application (certainly in all those we have considered), the inverse of the storage rule function  $\theta(S)$ , that is,  $\theta^{-1}(C)$ , is unique for  $C > 0$ . That is, the function  $\theta(S)$  is monotonically increasing for all values of  $S$  such that  $\theta(S) > 0$ . This means that each step in the procedure can be considerably simplified if, instead of finding for each possible value of  $S$  the value of  $C > 0$  which satisfies the stated condition, we find for each possible value of  $C > 0$  the value of  $S$  which satisfies the condition. The result is to obtain  $S$  as a function of  $C$ , and that function is the inverse of the optimal storage rule for the given year, say  $\hat{\theta}_t^{-1}(C)$ . To obtain the optimal storage rule  $\hat{\theta}_t(S)$ , we simply invert  $\hat{\theta}_t^{-1}(C)$ .

For the case of stationarity, the procedure is essentially the same, but the iterations are continued until the resulting  $\theta$  converges. That is, if  $\hat{\theta}(S)$  is the optimal stationary storage rule, then successive approximations  $\theta_0(S)$ ,  $\theta_1(S)$ ,  $\dots$ ,  $\theta_m(S)$ , such that  $\lim_{m \rightarrow \infty} \theta_m(S) = \hat{\theta}(S)$ ,

are obtained by letting  $\theta_0(S) = 0$  (or any positive constant or any monotonically increasing function) and (for  $m = 1, 2, \dots$ ) finding  $\theta_m(S)$  to satisfy the condition

$$\alpha E\rho[\theta_m(S) + x - \theta_{m-1}(S) + x] - \gamma'[\theta_m(S)] - \rho[S - \theta_m(S)] = 0 \quad (25)$$

for all values of  $S$ . Alternatively, and more simply, if  $\theta^{-1}(C)$  is unique for  $C > 0$ , the condition to be satisfied can be written

$$\alpha E\rho[C + x - \theta_{m-1}(C + x)] - \gamma'(C) - \rho[\theta_m^{-1}(C) - C] = 0 \quad (26)$$

for all values of  $C > 0$ .

The optimal stationary rule  $\hat{\theta}(S)$  is, then, the function  $\theta$  which satisfies the following equation for all values of  $S$ :

$$\alpha E\rho[\theta(S) + x - \theta(\theta(S) + x)] - \gamma'[\theta(S)] - \rho[S - \theta(S)] = 0 \quad (25.1)$$

Alternatively, if the optimal stationary rule  $\hat{\theta}(S)$  has a unique inverse,  $\hat{\theta}^{-1}(C)$ , for  $C > 0$ , then  $\hat{\theta}(S)$  is the function  $\theta$  which satisfies the following equation for all values of  $C > 0$ .

$$\alpha E\rho[C + x - \theta(C + x)] - \gamma'(C) - \rho[\theta^{-1}(C) - C] = 0 \quad (26.1)$$

For some purposes, it is convenient to rewrite the latter equation as:

$$\theta^{-1}(C) = C + \rho^{-1} \left\{ \alpha \int_0^\infty \rho[C + x - \theta(C + x)] dF(x) - \gamma'(C) \right\} \quad (26.2)$$

where  $\rho^{-1}$  is the inverse function of  $\rho$ , and the expectation operator is written out explicitly as the integral over the distribution  $F(x)$ .

value, it is necessary to integrate the whole function over the distribution of  $x_t$ , rather than simply to insert the expected value of  $x_t$ . That is, in general,

$$\begin{aligned} & E\rho_t[C + x_t - \theta_t(C + x_t)] \\ \text{is not equal to} & \rho_t[C + E x_t - \theta_t(C + E x_t)]. \end{aligned}$$

It may clarify matters to repeat, in slightly different form, the iterative procedure for finding the solution to equation (26.1), that is, the optimal stationary storage rule  $\hat{\theta}(S)$ , given the annual discount factor  $\alpha$  and the (stationary) functions marginal value  $\rho$ , marginal cost of storage  $\gamma'$ , and probability distribution of output  $F(x)$ . The following steps are involved:

1. Take  $\theta_0(S)=0$ , or, alternatively, an arbitrary function  $\theta_0(S)$  as the starting point.
2. Find  $\theta_1^{-1}(C)$  by

$$\theta_1^{-1}(C) = C + \rho^{-1} \left\{ \alpha \int_0^{\infty} \rho[C+x - \theta_0(C+x)] dF(x) - \gamma'(C) \right\} \quad (26.3)$$

Invert  $\theta_1^{-1}(C)$  to get  $\theta_1(S)$ .

3. In general, for  $m=1, 2, \dots$ , having found  $\theta_{m-1}(S)$ , find  $\theta_m^{-1}(C)$  by

$$\theta_m^{-1}(C) = C + \rho^{-1} \left\{ \alpha \int_0^{\infty} \rho[C+x - \theta_{m-1}(C+x)] dF(x) - \gamma'(C) \right\} \quad (26.4)$$

Invert  $\theta_m^{-1}(C)$  to get  $\theta_m(S)$ .

4. Then the optimal stationary storage rule is given by

$$\lim_{m \rightarrow \infty} \theta_m(S) = \hat{\theta}(S) \quad (27)$$

In computations for practical applications, the iterations are not, of course, continued to infinity, but only to the point where convergence is achieved; that is, to the point where  $\theta_m(S) = \theta_{m-1}(S)$ , for all relevant values of  $S$ . Once such convergence is obtained, further iterations in no way change the results.

*Computational considerations.*—The solution using the marginal value function is, of course, less general than that using the total value function, since the total value function must be differentiable so that the marginal value function exists. Furthermore, generalization of the solutions to include the possibility of nonindependence is usually easier if the total value function is used. However, for the case of independence, which has been assumed in all of the discussion so far, the solution and the computations using the marginal value function have several advantages over those using the total value function. One advantage is that, when using the marginal value function, the cumulative sums of discounted expected gains (that is, the  $\hat{V}_{1,n}$  functions) need not be computed at each step. Thus, computing labor is saved in each iteration, and further, the number of required iterations is less (in the case of stationarity), since the storage rule functions (the  $\theta$ 's) tend to converge more rapidly than do the corresponding  $\hat{V}_{1,n}$  functions. Additional characteristics of the solution and computations using the marginal value function are discussed in the remaining pages of this section.

The integrations over  $F(x)$  at each step, that is, the computations of the expected values  $E$ , still are carried out numerically, but values of the functions  $\rho$ ,  $\theta$  and  $\theta^{-1}$  can conveniently be found graphically. For the kinds of applications discussed beginning on page 21, a given number of iterations can be carried out in about one-fourth the number of computing man-hours required when using the total value

function, and with somewhat greater precision. This results because, when using the marginal value function method, interpolations can be made which cannot be used with the total value function method. For the applications which have been made, the number of iterations required to achieve convergence in  $\theta$ , within the limits of accuracy of the combined numerical-graphical procedure, varied from 7 to 15.

In cases where the number of required iterations is large, labor can be saved by a trial-and-error method as follows: We first define an operator  $\pi$  by

$$\pi\theta(C) = \alpha E\rho[C+x-\theta(C+x)] - \rho[\theta^{-1}(C) - C] \quad (28)$$

Then the optimal storage rule  $\hat{\theta}(S)$  is that function  $\theta$  which satisfies the equation

$$\pi\theta(C) = \gamma'(C) \quad (29)$$

Different  $\theta$ 's are tried, and the corresponding  $\pi\theta(C)$  functions computed, until a sufficiently close approximation to  $\gamma'(C)$  is obtained. After some practice, good approximations frequently can be obtained with relatively few trials. Once a fairly close approximation has been obtained, the corresponding  $\theta$  can be labeled  $\theta_0$ , and then the iterative procedure applied until complete convergence is attained, if desired.

Since numerical convergence, within the limits of computational accuracy, is not equivalent to mathematical convergence, it is desirable to be able to show the existence of an "upper bound" to the optimal stationary storage rule  $\hat{\theta}$ , that is, a function  $\theta_B$  (say) such that  $\theta_B(S) \geq \hat{\theta}(S)$  for every  $S$ . This can be readily done, using the marginal value function method. All that is required is to find a function  $\theta_B$  such that  $\pi\theta_B(C) < \gamma'(C)$  for every  $C$ ; it follows that  $\theta_B(S) \geq \hat{\theta}(S)$  for every  $S$ .<sup>22</sup>

## SPECIAL MATHEMATICAL RELATIONSHIPS OF INTEREST TO THE ECONOMIC ANALYST

*Relations between free-market and optimal governmental storage.*—From equation (26.1) on page 46, the following interesting equivalence relation can be shown: The amounts which would be stored under an optimal governmental storage program, that is, a program that maximizes the sum of discounted expected net gains to the general public, are exactly the same as the amounts which would be stored in the aggregate by private firms in a free market, if the following conditions are satisfied:

<sup>22</sup> This result is intuitively acceptable: uniformly lower storage costs imply uniformly higher optimal storage rules. The truth of this proposition also can be seen by setting  $\theta_B$  equal to  $\theta_0$  in the iterative procedure, and observing the relation between this  $\theta_0$  and the resulting  $\theta_1$ . Thus:

$$\alpha E\rho[C+x-\theta_0(C+x)] - \pi\theta_0(C) - \rho[\theta_0^{-1}(C) - C] = 0$$

and

$$\alpha E\rho[C+x-\theta_0(C+x)] - \gamma'(C) - \rho[\theta_1^{-1}(C) - C] = 0$$

Since  $\gamma'(C) < \pi\theta_0(C)$ , it follows that  $\rho[\theta_1^{-1}(C) - C] < \rho[\theta_0^{-1}(C) - C]$ , that is, since  $\rho$  is monotonically decreasing,  $\theta_1^{-1}(C) > \theta_0^{-1}(C)$ , so that,  $\theta_1(S) \leq \theta_0(S)$ .

1. The market is perfectly competitive, and all storing is done by firms seeking to maximize discounted expected profit;
2. The marginal value function  $\rho(Y)$  is the same as the market price function;
3. The market discount factor is the same as the Government's discount factor; and
4.  $\gamma'(C)$  is the price at which the amount  $C$  of storage space can be rented, that is, the "supply schedule" of storage space, minus the "marginal convenience benefit" of the amount  $C$  of stocks on hand.

Under these conditions, if  $\theta(S)$  is interpreted to mean the aggregate amount stored by private firms, the first term in equation (26.1) is, for any given year, the discounted expected price in the following year, the third term is the price in the given year, and the middle term is the per-unit marginal cost of storage. Only if  $\theta(S) = \hat{\theta}(S)$ , that is, only if the private firms' aggregate storage activity is such as to satisfy equation (26.1), is the market in equilibrium. If  $\theta(S) < \hat{\theta}(S)$  for some  $S$ , expected marginal returns are greater than marginal costs, and some firms tend to increase their amounts stored or to enter the storing business; conversely if  $\theta(S) > \hat{\theta}(S)$  for some  $S$ .

*Relations between the conditions and the optimal storage rule.*—Use of equation (26.1) also shows more clearly and simply than can otherwise be done the relationships between the conditions of the problem, that is, the discount factor  $\alpha$  and the functions marginal value  $\rho$ , marginal cost of storage  $\gamma'$ , and distribution of output  $F$ , and the solution to the problem, the optimal stationary storage rule  $\hat{\theta}$ . It can be shown fairly easily that certain kinds of changes in some of the conditions are equivalent, in their effects on the resulting optimal storage rule, to specified changes in other conditions. Equivalence relations of this kind, which are useful both for substantive and computational purposes, are illustrated in the following paragraphs.

For given  $\alpha$  and  $F$ , a change in  $\rho(Y)$  by a constant factor  $r$  is equivalent in its effects on  $\hat{\theta}$  to a change in  $\gamma'(C)$  by the constant factor  $1/r$ . From another viewpoint, a general price inflation or deflation which does not change the ratio of  $\rho(Y)$  to  $\gamma'(C)$  for any  $Y$  or  $C$ , and also does not change the interest rate or  $\alpha$ , has no effect on the optimal storage rule  $\hat{\theta}$ . Similarly, a change in  $\rho(Y)$  by the addition of a constant  $k$  is equivalent in its effects on  $\hat{\theta}$  to adding the constant  $(1-\alpha)k$  to  $\gamma'(C)$ . Also, for given  $\alpha$  and  $F$ , if  $\rho^*(Y) = r\rho(Y) + k$  and  $\gamma'^*(C) = (1/r)\gamma'(C) + (1/r)(1-\alpha)k$ , then the same  $\hat{\theta}$  which is optimal under  $\rho^*$ ,  $\gamma'^*$  is optimal under  $\rho$ ,  $\gamma'$ ; that is, a change of  $\rho$  to  $\rho^*$  is equivalent in its effects on  $\hat{\theta}$  to a change in  $\gamma'$  to  $\gamma'^*$ .<sup>23</sup>

Using the results of the last paragraph, it follows that if

$$\rho^*(Y) = r[\rho(Y) - P_0] + P_0 = r\rho(Y) + (1-r)P_0 \quad (30)$$

then a change in  $\rho$  to  $\rho^*$  is equivalent in its effects on  $\hat{\theta}$  to a change in  $\gamma'(C)$  to  $\gamma'^*(C)$ , where

$$\gamma'^*(C) = (1/r)\gamma'(C) + (1/r)(1-\alpha)(1-r)P_0 \quad (31)$$

<sup>23</sup> Proofs of these statements are given in Appendix Note 7.

The flexibility of the marginal value function is defined by

$$\epsilon(Y) = -[d\rho(Y)/dY] \cdot [Y/\rho(Y)] \quad (32)$$

That is, flexibility is the absolute value of the elasticity of marginal value with respect to quantity utilized. Let  $\epsilon_0$  be the value of the flexibility function at the point  $Y$  where  $\rho(Y) = P_0$ . Then changing  $\rho(Y)$  to  $\rho^*(Y)$ , as defined by equation (30), implies changing  $\epsilon_0$  to  $\epsilon_0^* = r\epsilon_0$ . Thus a change in the flexibility  $\epsilon_0$  by a factor  $r$ , where the change is accomplished by changing  $\rho$  as defined above, is equivalent in its effects on  $\hat{\theta}$  to changing  $\gamma'$  to

$$\gamma'^* = (1/r)\gamma' + (1/r)(1-\alpha)(1-r)P_0 \quad (33)$$

For a numerical illustration of the last result, suppose  $\epsilon_0 = 2.00$  and we wish to make  $\epsilon_0^* = 2.50$  by making

$$\rho^*(Y) = 1.25[\rho(Y) - P_0] + P_0 \quad (30.1)$$

where  $P_0 = \rho(\bar{X})$ , that is, the value of the marginal value function at  $Y = \text{the mean value of output } x$ . Suppose also that  $P_0 = \$1.50$  per bushel,  $\gamma'(C) = \$0.10$  per bushel (constant marginal cost of storage), and  $\alpha = 0.95$  (equivalent to an interest rate of about 5 percent per annum). A simple computation shows that changing  $\rho$  to  $\rho^*$  (and thereby changing  $\epsilon_0 = 2.00$  to  $\epsilon_0^* = 2.50$ ) is equivalent in its effects on  $\hat{\theta}$  to changing the marginal cost of storage from  $\gamma'(C) = \$0.10$  per bushel to  $\gamma'^*(C) = \$0.065$  per bushel.<sup>24</sup>

We next consider what can be said about the effects on the optimal storage rule of changing the variance of the probability distribution of output  $F(x)$ . To be specific, let  $F(x)$  be changed to  $G(x)$  by the relation

$$g[r(x-\mu)] = (1/r)f(x-\mu) \quad (34)$$

where  $f$  is the probability density function of the distribution  $F$ , transposed for convenience to take the origin at  $\mu$ , the mean of  $x$ ;  $g$  is the probability density function of the distribution  $G$ ; and  $r$  is a constant factor greater than zero.  $G$  then has the same general form and the same mean as  $F$ , but standard deviation  $\sigma_G = r\sigma_F$ . The problem is, for given  $\gamma'$ ,  $\alpha$  and  $\rho$ , to find a relation between the storage rule  $\hat{\theta}_G$  which is optimal under  $G$  and one which is optimal under  $F$ . The solution is to first find the rule  $\theta^*$  which is optimal under  $F$ ,  $\gamma'$ ,  $\alpha$ , and  $\rho^*$ , where  $\rho^*$  is defined by

$$\rho^*(Y-\mu) = \rho[r(Y-\mu)] \quad (35)$$

For linear  $\rho$ , this is equivalent to making

$$\rho^*(Y) = r[\rho(Y) - \rho(\mu)] + \rho(\mu) \quad (36)$$

Then the optimal rule under  $\gamma'$ ,  $\alpha$ ,  $G$  and  $\rho$  is <sup>25</sup>

$$\hat{\theta}_G(S-\mu) = r\theta^*[(1/r)(S-\mu)] \quad (37)$$

<sup>24</sup> If  $\rho(Y)$  has constant flexibility,  $\rho^*(Y)$  as defined here does not have constant flexibility, but if  $\rho(Y)$  is linear,  $\rho^*(Y)$  also is linear.

<sup>25</sup> See Appendix Note 8.

*Methods that allow for random fluctuations in demand.*—An important feature of the method that uses the marginal value function is that random fluctuations in the conditions of demand, as well as of supply, can readily be incorporated into the solution. The simplest case is one where the marginal value function  $\rho(Y)$  in each year (including the current year) is subject to the same probability distribution. One then could define  $R(Y) = E_{\rho}\rho(Y)$ , where  $E_{\rho}\rho(Y)$  is the integral of  $\rho(Y)$  over the probability distribution of  $\rho$ . Then  $\hat{\theta}$  is obtained as before, substituting  $R$  everywhere for  $\rho$ . However, it usually is more realistic to suppose that information about demand, or about the marginal value function, in the current year is better or more exact than the corresponding information for future years; that is, to treat information about demand in the same way as information about supply. If the current year's marginal value function is known and future years' marginal value functions are subject to known probability distributions, then an explicit solution would in general involve an iterative procedure similar to those already outlined, except that each step requires integration over the probability distribution of  $\rho$  as well as over  $F(x)$ .

By making a certain not unreasonable assumption about the way in which the random fluctuations in marginal value occur, the solution can be considerably simplified. The assumption is that the marginal value function in year  $t$  is given by  $\rho_t(Y_t + u_t)$ , where  $Y_t$  is quantity utilized, the value of  $u_t$  for the current year is known (designated  $U_1$ ), and  $u_t$  in each future year  $t$  is a random variable subject to some known or assumed probability distribution. If the function  $\rho$  is thought of as plotted on a graph with  $Y$  on the horizontal axis, the assumption is that the random fluctuations or shifts in the curve occur horizontally. This is analogous to assuming, for a demand schedule, that at a given price the quantity demanded is a random variable subject to a probability distribution, and that the probability distributions corresponding to different prices have different means but are otherwise identical.

With randomly fluctuating marginal value functions of the kind just described, the solution for the optimal storage rules  $\hat{\theta}$  is obtained as follows.<sup>26</sup> The storage rule  $\theta$  becomes a function not of  $S$  alone, but of  $S + U$ , that is,  $C = \theta(S + U)$  and  $\theta^{-1}(C) = S + U$ . From equation (26.1), the optimal storage rule  $\hat{\theta}$  is the function  $\theta$  which satisfies the equation

$$\alpha E_{x,u}[\rho[C + x + u - \theta(C + x + u)] - \gamma'(C) - \rho[\theta^{-1}(C) - C]] = 0 \quad (38)$$

where  $E_{x,u}$  means the integral over the probability distributions of  $x$  and  $u$ . A new random variable  $z = x + u$  can be defined, and its distribution determined from the distributions of  $x$  and  $u$ . Then the equation to be satisfied by  $\hat{\theta}$  can be written

$$\alpha E_{z,\rho}[C + z - \theta(C + z)] - \gamma'(C) - \rho[\theta^{-1}(C) - C] = 0 \quad (38.1)$$

<sup>26</sup> For simplicity, the discussion is for the case of stationarity, so the time subscripts are dropped; the modifications required for non-stationary should be clear to the reader.

$\hat{\theta}$  is computed by the procedure outlined earlier, keeping in mind that the resulting optimal storage rule is a function of  $S+U$  rather than  $S$  alone. Thus, if in the current year  $U=0$ , the only change in  $\hat{\theta}$  for the current year caused by the introduction of demand variability in future years is that due to the greater variability of  $z$  over that of  $x$ . If  $\rho[S-\theta(S)]$ , plotted against  $S$ , is concave upward, as it will be in most practical applications, the change in  $\hat{\theta}$  caused by introducing random variability in demand is upwards. That is, an optimal storage policy under conditions of random fluctuations in future demand calls for higher levels of storage than an optimal policy under conditions of fixed future demand schedules, other things being equal.

*Some computational aids.*—We next present mathematical proofs for the methods of obtaining approximate rules given in a preceding section and certain other devices by which the task of computing optimal storage rules under specified conditions, using the marginal value function, can be somewhat lightened. Some of the relationships discussed also are of interest in themselves. Most of the discussion is, for simplicity, in terms of finding the optimal storage rule for the case of stationarity and with no random fluctuations in marginal value or demand, but some of the ideas also can be applied, with suitable modification, to the cases of non-stationarity and random fluctuations in demand.

If the marginal value function  $\rho(Y)$  is linear, say  $\rho(Y)=q-pY$ , where  $q$  and  $p$  are constants, then equation (26.1) (see p. 46) reduces to:

$$\theta^{-1}(C)=\gamma'(C)/p+(1-\alpha)q/p+\alpha\mu+(1+\alpha)C-\alpha E\theta(C+x) \quad (39)$$

where  $\mu$  is  $Ex$ , the mean of the probability distribution of  $x$ . If the marginal cost of storage  $\gamma'(C)$  is constant, designated  $\gamma'$ , equation (39) can be written

$$\theta^{-1}(C)=K_1+(1+\alpha)C-\alpha \int_{k-C}^{\infty} \theta(C+x)f(x)dx \quad (40)$$

where  $K_1$  is a constant,  $\gamma'/p+(1-\alpha)q/p+\alpha\mu$ ;  $k$  is the value of  $S$  (to be determined, along with the rest of the storage rule) such that for  $S \leq k$ ,  $\theta(S)=0$ ;  $f(x)$  is the probability density of  $x$ ; and  $E\theta(C+x)$  is written as the integral to emphasize that the integration is not taken over the full range of  $F(x)$ .

Equations (39) or (40) indicate that, even if the marginal value function  $\rho(Y)$  and the marginal cost of storage function  $\gamma'(C)$  are linear, the optimal storage rule  $\hat{\theta}(S)$  cannot be linear, even over the range  $S > k$ . The solution  $\hat{\theta}$  is obtained by iteration as before, but the computations become somewhat simpler, since no computation of values of the function  $\rho$  (graphical or otherwise) are required. At each step, having obtained values of the function  $\theta_m^{-1}(C)$ , values of  $\theta_m(S)$  in most applications can be obtained numerically by linear interpolation. Even though  $\hat{\theta}(S)$  is not linear, in most applications it is sufficiently close to being linear so that linear interpolations over narrow ranges give adequate accuracy.

Equation (40) indicates that, for given  $\alpha$  and  $F$ , any changes in  $\rho$  and  $\gamma'$  which leave the constant  $K_2 = \gamma'/p + (1-\alpha)q/p$  unchanged also leave the optimal storage rule  $\hat{\theta}$  unchanged. For all  $\rho$ 's which pass through the point  $(P_0, Y_0)$ ,  $q/p = P_0/p + Y_0$ , so that any changes in  $\gamma'$  and  $p$  which leave the constant  $K_3 = \gamma'/p + (1-\alpha)P_0/p$  unchanged also leave  $\hat{\theta}$  unchanged. But a change in  $p$  by a factor  $r$  is equivalent to changing  $\epsilon_0$  (the flexibility at the point  $P_0, Y_0$ ) by the same factor  $r$ . So equivalence relations between changes in  $\epsilon_0$  and changes in  $\gamma'$  can be obtained directly for the linear- $\rho$  case, and they are, of course, the same as those obtained on p. 49 for the more general case.

As pointed out on p. 33, in many applications which have been carried out to date, the optimal storage rule  $\hat{\theta}(S)$ , when computed for a given set of conditions and plotted on a graph with  $S$  on the horizontal axis, is a curve approximately "parallel" to and lying to the left of, an optimal storage rule, say  $\theta^0(S)$ , which is computed using the same set of conditions except that output variability in future years is assumed to be zero and output in each year is taken equal to the expected value or  $Ex$ . That is,  $\hat{\theta}(S) \approx \theta^0(S+d)$ , where  $d$  is some constant.

The computation required to obtain the optimal rule  $\theta^0(S)$  under the assumption of zero variability in future outputs is a relatively simple one. The optimal rule,  $C = \theta^0(S)$ , may be graphed in a series of monotonically increasing connected line segments, as in figure 7.

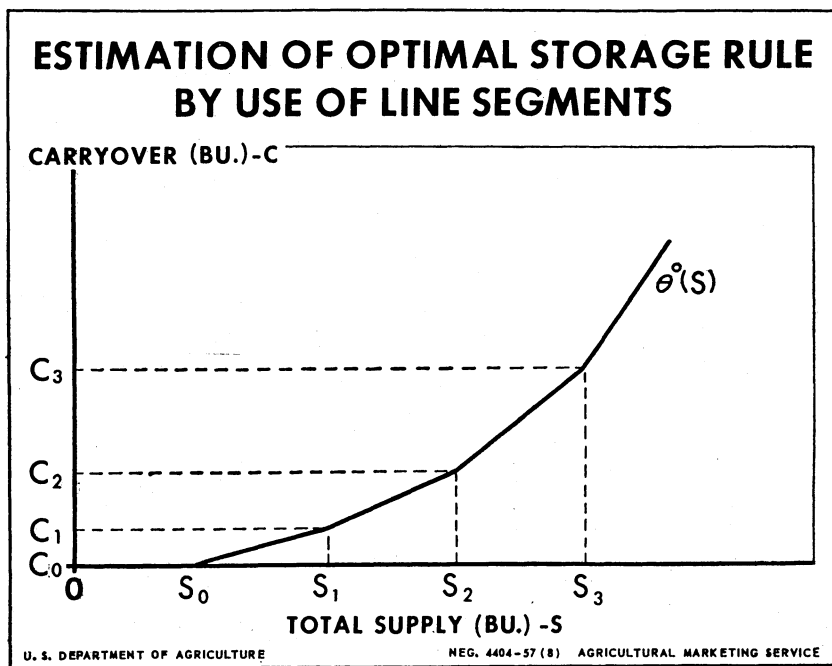


FIGURE 7.—Coordinates for the connecting points of the line segments in this chart are obtained by formulas (41) to (44). The monotonically increasing segments suggest the general shape of the optimal storage rule under the conditions specified.

If  $\rho(Y)$  is linear, the segments of  $\theta^0(S)$  are also linear; if  $\rho(Y)$  is not linear, the segments are not linear, but may be adequately approximated for most practical purposes by linear segments. Hence all that is required to determine the rule  $\theta^0(S)$  is to determine the connecting points of the segments, which are designated in the chart as  $(S_i, C_i)$  ( $i=0, 1, 2, \dots$ ). The subscripts  $i$  on  $S$  and  $C$  do not here represent years, of course, but simply the different points along the rule  $C=\theta^0(S)$ . Let  $h$  be the harvest in each future year; if the rule is being calculated for purposes of approximating the optimal rule with variable yields,  $h=\mu=\bar{E}x$ . As before,  $\rho(Y)$  is the marginal value function,  $\alpha$  is the discount factor, and  $\gamma'$  is the marginal cost of storage (here assumed constant).

We define a linear operator  $D$  operating on a variable  $Z$  by  $DZ=\alpha Z-\gamma'$ . Then the optimal rule  $\theta^0(S)$ , for the case of constant harvest in future years, is determined by obtaining the segment connecting points  $(S_i, C_i)$  as follows, where  $M_h$  is the value of the marginal social value function at  $Y=h$ , that is,  $M_h=\rho(h)$ :

$$S_0=\rho^{-1}(DM_h) \quad (41)$$

$$S_i=S_{i-1}-h+\rho^{-1}(D^{i+1}M_h) \quad (i=1, 2, \dots) \quad (42)$$

$$C_0=0 \quad (43)$$

$$C_i=S_{i-1}-h \quad (44)$$

$$=S_i-\rho^{-1}(D^{i+1}M_h) \quad (44.1)$$

$$=C_{i-1}-h+\rho^{-1}(D^i M_h) \quad (i=1, 2, \dots) \quad (44.2)$$

where the superscript  $(-1)$  on  $\rho$  indicates the inverse of the function, and the superscript  $(i+1)$  on  $D$  indicates that the operation is to be performed  $i+1$  times. The three expressions for  $C_i$  are equivalent; all are given to indicate the inter-relationships involved and to give the computer a choice.  $S_0$  is the  $S$ -axis intercept of  $\theta^0(S)$ . The meaning of the  $D$  operations may be clarified by noting that:

$$\text{For } i=1, \quad D^{i+1}M_h=D^2M_h=\alpha(\alpha M_h-\gamma')-\gamma' \quad (45)$$

$$\text{For } i=2, \quad D^{i+1}M_h=D^3M_h=\alpha[\alpha(\alpha M_h-\gamma')-\gamma']-\gamma' \quad (45.1)$$

In general,

$$D^{i+1}M_h=\alpha^{i+1}M_h-\gamma'\sum_{j=0}^i\alpha^j \quad (45.2)$$

Proofs of these results are given in Appendix note 9. A numerical illustration was given on pp. 34-36.

We next consider the problem of determining an approximate value of  $k$ , the  $S$ -axis intercept of the optimal storage rule  $\hat{\theta}(S)$  for the case of uncertainty in future outputs, that is, when the harvest  $x$  in any future year is a random variable subject to an estimated probability distribution. The exact value of  $k$  is, of course, obtained from the iterative computation procedure along with the rest of the storage rule. The approximation considered here is for the purpose of obtaining an approximate rule, by the method given starting at the bot-

tom of page 33. The approximate value, say  $k_*$ , is obtained by solving the following equation for  $k$ :

$$k = M - \alpha \theta'_s L(k) \quad (13.1)$$

where the symbols are defined as follows:

$M$  is a constant that equals

$$\mu - (1 - \alpha) \rho(\mu) / \rho'(\mu) - \gamma' / \rho'(\mu) \quad (12.1)$$

where  $\mu$  is the mean of the probability distribution of  $x$ ,  $\rho(\mu)$  is the value of  $\rho(Y)$  at  $Y = \mu$ ,  $\rho'(\mu)$  is the value of the slope of  $\rho(Y)$  at  $Y = \mu$ , and  $\alpha$  and  $\gamma'$  are the annual discount factor and the marginal cost of storage respectively, as before.

$\theta'_s$  is an advance estimate or approximation of the average slope of the optimal storage rule  $\hat{\theta}(S)$ . This approximation can be obtained from the slope of  $\theta^0(S)$  when the conditions other than yield variability are the same as those of the rule now being approximated (see p. 38).

$L(k)$  is the function defined by  $L(k) = \int_k^\infty (x - k) dF(x)$ . The values of this function for different values of  $k$  in most applications must be computed numerically. Then the value of  $k$  which comes close to satisfying the equation  $k = M - \alpha \theta'_s L(k)$  can be obtained by linear interpolation, giving the desired approximate value  $k_*$ . The function  $L(k)$  depends only on the probability distribution  $F(x)$ , and once obtained for a particular  $F$  can be used for different sets of assumptions about the other conditions. Derivations of these results are given in Appendix note 10. A numerical illustration was given on pages 36-39.

*Expected returns to storage.*—The "expected returns to storage" obtained by following an optimal storage policy for an  $n$ -year period may be defined as the difference between the sum of discounted expected gains when the optimal policy is followed and the sum of discounted expected gains when the carryover in every year is zero. If the storage rule is computed using the total value function, this difference can be obtained directly from the results of such computations, because of the fact that the maximized sum of discounted expected gains is computed at each step. Thus, for the case of non-stationarity, the expected returns to storage for the  $n$ -year period may be written as a function of the initial year's total supply as follows:

$$R_n(S_1) = \hat{V}_{1,n}(S_1) - \delta_1(S_1) - \sum_{t=2}^n \alpha^{t-1} \int_0^\infty \delta_t(x_t) dF_t(x_t) \quad (46)$$

where  $\hat{V}_{1,n}(S_1)$  is the (maximized) sum of discounted expected gains under the optimal storage policy and the other two terms are the sum of discounted expected gains when the carryover in every year is zero.

For the case of stationarity, the corresponding expression for  $n$  years, dropping the subscript 1 on  $S_1$  is:

$$R_n(S) = J^{n-1} \delta(S) - \delta(S) - \left( \sum_{t=2}^n \alpha^{t-1} \right) \int_0^\infty \delta(x) dF(x) \quad (47)$$

and for all future years is:

$$R_{\infty}(S) = \lim_{n \rightarrow \infty} R_n(S) = \beta(S) - \delta(S) - [\alpha/(1-\alpha)] \int_0^{\infty} \delta(x) dF(x) \quad (47.1)$$

In each case the expected returns are a function of the initial year's total supply  $S$ . Calculation of such expected returns functions gives one a measure of the economic importance of a storage policy. Expected gains and losses under alternative (non-optimal but non-zero) storage rules also may be computed. These aid in determining the economic costs of adopting such non-optimal policies instead of an optimal policy.

Calculation of the expected returns to storage is not quite so straightforward if the optimal storage rules have been computed using the marginal value function. However, for the case of stationarity, the problem still can be solved fairly simply. The question is, having found the optimal stationary storage rule  $\hat{\theta}$  using the  $\rho$ -function, instead of finding the maximized sum of discounted expected gains  $\beta(S)$  directly using the  $\delta$ -function, is it now possible to find  $\beta(S)$  from  $\hat{\theta}(S)$ ? The answer is yes, as follows: We know from the proof of the equivalence of the two methods of solution in Appendix note 6 that

$$d\beta(S)/dS = \rho[S - \hat{\theta}(S)] \quad (48)$$

Therefore,

$$\beta(S) = \int_0^S \rho[z - \hat{\theta}(z)] dz + K \quad (49)$$

where the first term to the right of the equality sign is a function of  $S$  (call it  $\lambda(S)$ ) which can be determined from  $\hat{\theta}$ , and  $K$  is a constant. The value of  $K$  is found as follows: From equation (22),  $J\beta(S) = \beta(S)$ , that is,

$$\delta[S - \hat{\theta}(S)] - \gamma[\hat{\theta}(S)] + \alpha \int_0^{\infty} \beta[\hat{\theta}(S) + x] dF(x) = \beta(S) \quad (50)$$

Substituting  $\beta(S) = \lambda(S) + K$  in this equation gives

$$(1-\alpha)K = \delta[S - \hat{\theta}(S)] - \gamma[\hat{\theta}(S)] + \alpha \int_0^{\infty} \lambda[\hat{\theta}(S) + x] dF(x) - \lambda(S) \quad (50.1)$$

It can be verified easily that the expression on the right of the equality sign is a constant. Some results of calculating expected returns to storage for specific storage rules are given in Appendix note 2.

*The equilibrium level of storage.*—The "equilibrium level" of carry-over is defined in the following way. Under stationarity, if the same storage rule  $\theta(S)$  is applied every year, whether  $\theta$  is optimal or not, and if  $\theta$  fulfills the following conditions: (1)  $\theta(S) < S$  for all  $S$ , (2)  $\theta$  is continuous and  $0 \leq d\theta(S)/dS < 1$ , and (3)  $\theta(x_{\max}) > 0$ , where  $x_{\max}$  is the

greatest possible value of  $x$ , then the following statements can be shown to be true (see Appendix note 11):

1. There exists a value  $C^* > 0$ , such that  $\int_0^\infty \theta(C^* + x) dF(x) = C^*$ ; that is, if the carryover in year  $t$ ,  $C_t$ , equals  $C^*$ , the expected carryover in year  $t+1$ ,  $EC_{t+1}$  also equals  $C^*$ .

2.  $C^*$  is unique.

3. For any  $C_t$  not equal to  $C^*$ ,  $EC_{t+1}$  is between  $C_t$  and  $C^*$ .

The value of  $C^*$  can be found readily, for a given  $\theta$ , by trial and error. Its chief uses are

1. To enable the economic analyst to make quick comparisons among the effects on "average" carryover levels of different assumptions about the conditions  $\alpha$ ,  $\gamma$ ,  $F$  and  $\rho$  (or  $\delta$ ) and the resulting storage rules. Thus, instead of comparing two rules in entirety by use of a graph or a table of values, one can compare the two resulting equilibrium levels. This does not, of course, give a complete picture of the effective differences in the two rules.

2. To enable the analyst to make rough comparisons between "average" carryover levels that result under optimal storage rules satisfying the criteria specified in this handbook and carryover levels recommended by other writers or to satisfy other criteria.

## METHODS THAT ALLOW FOR CONTINGENCIES

Optimal carryover rules can be computed in various ways for a period in which the nation faces the possibility of the future occurrence of war or other disaster with similar consequences if the probability of such an occurrence can be estimated and the effects of such an occurrence on the relevant conditions (demand, storage cost, interest rate, and output) also can be estimated.

For example, if (1) the probability of the nation's being at war in any future year is  $\beta$ , so that the probability of peace is  $1-\beta$ ; (2) the marginal value function under war conditions,  $\rho_w$ , is related to that for peace,  $\rho$ , by  $\rho_w(Y+U) = \rho(Y)$  where  $U$  is a known constant; and (3) the other conditions ( $\gamma$ ,  $\alpha$ , and  $F$ ) are unaffected by war, then the method outlined on page 51 for the case involving this particular kind of random variation in  $\rho$  can be used.

*A suggested approach.*—It seems unrealistic to assume that the probability that a state of war exists is independent from one year to the next. An assumption that may conform better with experience is to say that the probability of a war *starting* in any future year is  $\beta$ . We then can compute optimal carryover rules for the years of peace (that is, for the period of defense preparation) if we know or can assume the carryover rule for the first year of war ( $\theta_w$ , say).

$\theta_w$  could be assumed directly or, perhaps better, computed on the basis of assumptions about the expected duration of the war and the changes caused by the war in  $\alpha$ ,  $\gamma$ ,  $\rho$  and  $F$ . For example, if the war is expected to continue indefinitely and to cause no changes in  $\alpha$ ,  $\gamma'$  and  $F$  but to cause demand to increase by the amount  $U$  for any price, then from equation (26.1) the optimal  $\theta_w$  must satisfy

$$\alpha E \rho_w[C+x-\theta_w(C+x)] - \gamma'(C) - \rho_w[\theta_w^{-1}(C) - C] = 0 \quad (26.5)$$

where  $\rho_w(Y) = \rho(Y-U)$ .

Having found or assumed  $\theta_w$ , we can find the optimal storage rule  $\hat{\theta}_d$  for the years of war-preparedness, that is, the rule which will maximize the sum of discounted expected gains during those years, by finding that  $\hat{\theta}_d$  which satisfies

$$\alpha\beta E\rho_w[C+x-\theta_w(C+x)] + \alpha(1-\beta)E\rho[C+x-\hat{\theta}_d(C+x)] - \gamma'(C) - \rho[\hat{\theta}_d^{-1}(C) - C] = 0 \quad (51)$$

The first term is a determinable function of  $C$ , so the method of solving for  $\hat{\theta}_d$  is essentially the same as that outlined in pp. 44-47 and used in the applications to feed grains.

Once the optimal war-preparatory  $\hat{\theta}_d$  is determined, the corresponding equilibrium level  $C_d$  (say) and the equilibrium level  $C^*$  that results under the optimal rule with a probability of war equal to zero can be found. Then, if one likes, the difference between  $C_d$  and  $C^*$  can be considered as a "war reserve." However, it should be emphasized that this is not a separate stock. The primary effect of introducing the war contingency is a change in the storage rule itself; the change in equilibrium level of carryover is simply a concomitant effect.

*An application.*—Computations of explicit war-preparatory rules for sets of conditions corresponding to those used for the rules given in table 1 have not been carried out. However, an idea of the effect of allowing for war contingencies on storage policy, under such conditions, can be obtained as follows:

Assume that:

- (1) The probability of a war starting in any future year is  $\beta=0.2$ ;
- (2) During the war, the quantity demanded at any given price is 4.5 bushels per acre greater than in peacetime, that is,  $\rho_w(Y) = \rho(Y-4.5)$ ;
- (3) In peacetime  $\rho$  is the same as for  $\hat{\theta}_1$  in table 1, that is,  $\rho(Y) = 4.50 - 0.10(Y)$ , and in both peace and war,  $\alpha$ ,  $\gamma'$  and  $F$  are the same as for  $\hat{\theta}_1$  ( $\alpha=0.95$ ,  $\gamma'=0.10$ ,  $\sigma=3.03$ ).

Then  $\rho_w(Y) = \rho(Y) + 0.45$ , so (utilizing the results on page 49) if the war is assumed to go on forever,  $\theta_w$  under  $\rho_w$ ,  $\gamma'$  is equivalent to the  $\theta$  that is optimal under  $\rho$ ,  $\gamma'^*$ , where  $\gamma'^*=0.1225$ . This implies that  $\theta_w$  is slightly lower than  $\hat{\theta}_1$ , so we get a higher war-preparatory rule  $\hat{\theta}_d$  than is actually optimal by taking  $\theta_w = \hat{\theta}_1$ . If the war is not assumed to go on forever,  $\theta_w$  would be still lower.

From equation (51), under the conditions stipulated,

$$E\rho_w[C+x-\theta_w(C+x)] = E\rho[C+x-\hat{\theta}_1(C+x)] + pU \quad (52)$$

$$= q - pC - p\mu + p \int_{k_1-C}^{\infty} \hat{\theta}_1(C+x) dF(x) + pU \quad (52.1)$$

where  $k_1$  is the value of  $S$  below which  $\hat{\theta}_1(S)=0$ , and  $U=4.5$ . Also,

$$E\rho[C+x-\hat{\theta}_d(C+x)] = q - pC - p\mu + p \int_{k_d-C}^{\infty} \hat{\theta}_d(C+x) dF(x) \quad (53)$$

so equation (51) becomes

$$\alpha\beta pU + \alpha\beta p \int_{k_1-C}^{\infty} \hat{\theta}_1(C+x)dF(x) - \alpha\beta p \int_{k_d-C}^{\infty} \hat{\theta}_d(C+x)dF(x) + \\ \alpha E\rho[C+x-\hat{\theta}_d(C+x)] - \gamma'(C) - \rho[\hat{\theta}_d^{-1}(C) - C] = 0 \quad (54)$$

Finally, following the form of equation (39) (page 52), we obtain

$$\hat{\theta}_d^{-1}(C) = \gamma'/p + (1-\alpha)q/p + \alpha\mu - \alpha\beta U + (1+\alpha)C - \\ \alpha\beta \int_{k_1-C}^{\infty} \hat{\theta}_1(C+x)dF(x) - \alpha(1-\beta) \int_{k_d-C}^{\infty} \hat{\theta}_d(C+x)dF(x) \quad (55)$$

Since  $\hat{\theta}_1 < \hat{\theta}_d$ , we again get a  $\hat{\theta}_d$  which is higher than it should be by substituting  $\hat{\theta}_d$  for  $\hat{\theta}_1$ . This gives us

$$\hat{\theta}_d^{-1}(C) = \gamma'/p + (1-\alpha)q/p + \alpha\mu - \alpha\beta U + (1+\alpha)C - \alpha \int_{k_d-C}^{\infty} \hat{\theta}_d(C+x)dF(x) \quad (56)$$

Equation (56) is similar to equations (39) and (40) (page 52) for  $\hat{\theta}_1$ , except that instead of  $K_1=31.24$  ( $K_1$  is defined on page 52), we have  $K_1 - \alpha\beta U = 31.24 - 0.86 = 30.38$ . Comparing this value with the values of  $K_1$  corresponding to the conditions of  $\hat{\theta}_7$ ,  $\hat{\theta}_3$  and  $\hat{\theta}_5$  of 30.89, 30.64, and 30.54, respectively (see table 1), we find that our "conservative"  $\hat{\theta}_d$  is somewhat higher than  $\hat{\theta}_5$ , but not as high as  $\hat{\theta}_4$ .

If we take  $\hat{\theta}_5$  as an approximate  $\hat{\theta}_d$ , the "war reserve" is 56 million bushels  $[(0.7-0.3) \times 140 \text{ million}]$ . If  $\hat{\theta}_4$  is used as a doubly conservative approximation to  $\hat{\theta}_d$ , the war reserve is 154 million bushels.

## SOLUTIONS THAT ALLOW FOR LAG EFFECTS IN THE CONDITIONS

On page 15, methods are discussed by which the effect of a change in one year's supply of grain on the following year's livestock inventory can be allowed for, at least approximately, by an appropriate adjustment in the marginal value function. However, certain other kinds of lag effects may be more difficult to handle. If such effects can be quantified, the total value function for a given year  $t$  can be written as a function of both the quantity utilized in year  $t$  and the quantity utilized in the preceding year,  $t-1$ , that is,  $\delta_t = \delta_t(Y_t, Y_{t-1})$ . The optimal storage rule for a given year  $t$  then becomes a function of both total supply  $S$  in that year and the quantity utilized in the preceding year,  $\hat{\theta}_t(S_t, Y_{t-1})$ , or, in the case of stationarity,  $\hat{\theta}(S, Y_{-1})$ , where  $Y_{-1}$  is the quantity utilized in the year preceding the application of the rule. The solution may be written out explicitly for the case of stationarity as follows:

$$\hat{V}_{n,n} = \text{Max}_{0 \leq C \leq S} [\delta(S-C, Y_{-1}) - \gamma(C)] = \hat{V}_{n,n}(S, Y_{-1}) \quad (57)$$

and  $\hat{\theta}_n(S, Y_{-1})$  = the value of  $C$  that achieves the maximization.

$$\hat{V}_{n-1,n} = \text{Max}_{0 \leq C \leq S} [\delta(S-C, Y_{-1}) - \gamma(C) + \alpha E \hat{V}_{n,n}(C+x, S-C)] = \hat{V}_{n-1,n}(S, Y_{-1}) \quad (57.1)$$

and  $\hat{\theta}_{n-1}(S, Y_{-1})$  = the value of  $C$  that achieves the maximization.

We continue until  $\hat{V}_{2,n}(S, Y_{-1})$  is obtained; then

$$\hat{V}_{1,n} = \text{Max}_{0 \leq C \leq S} [\delta(S-C, Y_{-1}) - \gamma(C) + \alpha E \hat{V}_{2,n}(C+x, S-C)] = \hat{V}_{1,n}(S, Y_{-1}) \quad (57.2)$$

and  $\hat{\theta}_1(S, Y_{-1})$  = the value of  $C$  that achieves the maximization. For the last step,  $Y_{-1} = Y_0$ , the quantity consumed in the year preceding the initial year of the program.

The solution can be summarized more concisely by defining the operator  $J$  as

$$J\phi(S, Y_{-1}) = \text{Max}_{0 \leq C \leq S} [\delta(S-C, Y_{-1}) - \gamma(C) + \alpha E \phi(C+x, S-C)] \quad (58)$$

Then,

$$\hat{V}_{1,n} = J^{n-1} \delta(S, Y_{-1}) \quad (57.3)$$

Computations required are, of course, considerably more voluminous than in cases employing functions of one argument. The modifications required in the outline to allow for non-stationarity should be clear. Solutions that allow for lags in the other functions, that is, in the cost of storage and the distributions of output, can be obtained in an analogous way.<sup>27</sup>

## OPTIMAL MULTIREGIONAL STORAGE RULES MATHEMATICAL SOLUTIONS

Suppose we have  $m$  regions for which the following are known:

- (1) Total value functions:  $\delta_1(Y_1), \dots, \delta_m(Y_m)$
- (2) Cost of storage functions:  $\gamma_1(C_1), \dots, \gamma_m(C_m)$
- (3) Cost of transport functions:  $\tau_{ij}(Q_{ij}), i, j = 1, \dots, m$
- (4) Probability distribution of outputs:  $F(x_1, \dots, x_m)$

The subscripts refer to regions, not years. The solution is written only for the case of stationarity, so the year need not be indicated explicitly. Thus,  $\delta_i(Y_i)$  is the total value of quantity  $Y_i$  consumed in region  $i$  in a given year;  $\gamma_i(C_i)$  is the storage cost of carrying over the quantity  $C_i$  in region  $i$  in a given year; and  $x_i$  is the quantity produced in region  $i$  in a given year.  $Q_{ij}$  is the amount transported from region  $i$  to region  $j$ , and  $\tau_{ij}(Q_{ij})$  is the cost of that transport.

<sup>27</sup> A solution that incorporates first-order serial dependence in the distributions of yields, applied to compute optimal storage rules for wheat, is given in an unpublished manuscript by R. L. Gustafson entitled "The Storage of Grains to Offset Fluctuations in Yields."

*Based on the total value function.*—The total gain in a given year, for the nation as a whole, is defined as:

$$W = \sum_{i=1}^m \delta_i(Y_i) - \sum_{i=1}^m \gamma_i(C_i) - \sum_{i,j} \tau_{ij}(Q_{ij}) \quad (59)$$

Thus the individual regional gains are assumed to be additive to get the gain for the entire economy. The problem is, given the initial supplies  $S_1, \dots, S_m$ , to find the storage rules  $\hat{\theta}_1, \dots, \hat{\theta}_m$  which maximize the sum of discounted expected gains over some  $n$ -year period, or, in the limit, over all future years.

Let  $Z_i = S_i - C_i$ , so that  $Y_i = Z_i - Q_i$ , where  $Q_i = \sum_{j=1}^m Q_{ij}$  is the total amount transported out of region  $i$ . Let  $Q$  be the vector  $(Q_1, \dots, Q_m)$ . We define the function  $\lambda$  as follows:

$$\lambda(Z_1, \dots, Z_m) = \text{Max}_Q \left[ \sum_i \delta_i(Z_i - Q_i) - \sum_{i,j} \tau_{ij}(Q_{ij}) \right] \quad (60)$$

The problem of finding  $Q$  to get the value of  $\lambda$  is exactly the same as the maximization of "social pay-off" as discussed by Samuelson (9), provided the  $\delta$ 's are defined as areas under the demand curves. Also, as Samuelson demonstrates, this maximization problem is equivalent to the inter-spatial equilibrium problem for a free market. In other words, just as we have shown the equivalence of the conditions for inter-temporal equilibrium in a free market and the conditions for the maximization of net gain to the general public (see page 48), so Samuelson shows the equivalence of the conditions for inter-spatial equilibrium in a free market and the conditions for the maximization of net gain to the nation as a whole, where total value is taken to be the integral of the market price function. Here we are concerned with maximizing net gains both inter-spatially and inter-temporally.

We define the operator  $J$  as follows:

$$J\phi(S_1, \dots, S_m) = \text{Max}_O \left[ \lambda(S_1 - C_1, \dots, S_m - C_m) - \sum_i \gamma_i(C_i) + \alpha E\phi(C_1 + x_1, \dots, C_m + x_m) \right] \quad (61)$$

where  $C$  is the vector  $(C_1, \dots, C_m)$  and  $\text{Max}_O$  means the maximum with respect to  $C$ , subject to the restrictions  $0 \leq \sum_i C_i \leq \sum_i S_i$  and  $C_i \geq 0$  for all  $i$ .

We now write down the solution as follows:

$$\hat{V}_{1,1} = \text{Max}_O \left[ \lambda(S_1 - C_1, \dots, S_m - C_m) - \sum_i \gamma_i(C_i) \right] = \lambda(S_1, \dots, S_m) \quad (62)$$

$$\begin{aligned} \hat{V}_{1,2} = & \text{Max}_O \left[ \lambda(S_1 - C_1, \dots, S_m - C_m) - \right. \\ & \left. \sum_i \gamma_i(C_i) + \alpha E\lambda(C_1 + x_1, \dots, C_m + x_m) \right] \end{aligned} \quad (63)$$

$$= J\lambda(S_1, \dots, S_m) \quad (63.1)$$

$$\hat{V}_{1,3} = \underset{0}{\text{Max}}[\lambda(S_1 - C_1, \dots, S_m - C_m) - \sum_i \gamma_i(C_i) + \alpha E J \lambda(C_1 + x_1, \dots, C_m + x_m)] \quad (64)$$

$$= J^2 \lambda(S_1, \dots, S_m) \quad (64.1)$$

and in general,

$$\hat{V}_{1,n} = \underset{0}{\text{Max}}[\lambda(S_1 - C_1, \dots, S_m - C_m) - \sum_i \gamma_i(C_i) + \alpha E J^{n-2} \lambda(C_1 + x_1, \dots, C_m + x_m)] \quad (65)$$

$$= J^{n-1} \lambda(S_1, \dots, S_m) \quad (65.1)$$

The carryover in each region,  $C_i$ , or the storage rule for each region,  $\hat{\theta}_i$ , thus becomes a function of the supplies in all the regions ( $S_1, \dots, S_m$ ).

To indicate the extent to which a multi-regional solution magnifies the computational requirements, we estimate that, for cases similar to the applications discussed on pages 21-32, going from a 1-region to a 2-region solution increases the number of computational operations by a factor of about 200, and going from a 1-region to a 3-region solution increases the number of computational operations by a factor of about 40,000.

*Based on the marginal value function.*—We next consider the use of marginal value (or price) functions,  $\rho$ . For given  $\rho_1, \dots, \rho_m$ , the price in any region  $i$  in any year is, under spatial equilibrium, a function of (1) ( $Z_1, \dots, Z_m$ ), where  $Z_i = S_i - C_i$ , and (2) the costs of transport  $\tau_{ij}$  ( $i, j = 1, \dots, m$ ). That is, for given  $\tau_{ij}$ ,  $\rho_i = \psi_i(Z_1, \dots, Z_m)$ . As shown by Samuelson (9), the functions  $\psi_1, \dots, \psi_m$  can, with some effort, be determined. If we wish to maximize both inter-temporal and inter-spatial gain, we must find regional storage rules  $\hat{\theta}_i$  ( $i = 1, \dots, m$ ) each of which is a function of  $S_1, \dots, S_m$ .

Thus, for a 2-year period ( $n=2$ ), we find for each set of values ( $S_1, \dots, S_m$ ) the values of  $C_1, \dots, C_m$  such that (for  $i=1, \dots, m$ )

$$\alpha E \psi_i(C_1 + x_1, \dots, C_m + x_m) - \psi_i(S_1 - C_1, \dots, S_m - C_m) - \gamma'_i(C_i) = 0 \quad (66)$$

This gives  $\theta_{11}(S_1, \dots, S_m)$ ,  $i=1, \dots, m$ .

For  $n=3$ , we find for each set of values ( $S_1, \dots, S_m$ ) the values of  $C_1, \dots, C_m$  such that (for  $i=1, \dots, m$ )

$$\begin{aligned} \alpha E \psi_i[C_1 + x_1 - \theta_{11}(C_1 + x_1, \dots, C_m + x_m), \dots, \\ C_m + x_m - \theta_{m1}(C_1 + x_1, \dots, C_m + x_m)] - \\ \psi_i(S_1 - C_1, \dots, S_m - C_m) - \gamma'_i(C_i) = 0 \end{aligned} \quad (67)$$

This gives  $\theta_{12}(S_1, \dots, S_m)$ ,  $i=1, \dots, m$ .

The procedure may conceptually be continued until convergence is reached, that is, until  $\theta_{i,n} = \theta_{i,n-1} = \hat{\theta}_i$ ,  $i=1, \dots, m$ . The computations, however, are formidable, even for the simplest case, that is, a 2-year, 2-region model, as may be seen from the following example.

### AN EXAMPLE FOR TWO REGIONS AND TWO YEARS

Suppose each of two regions has the same  $\rho$ ,  $F$  and  $\gamma$ , with the  $F$ 's independent, and  $\tau_{12} = \tau_{21} = \tau$  and  $\rho(Y) = q - pY$ . Then the price in region 1,  $\psi_1$ , is given by one of the following:

(1) If  $Y_1 > Y_2 + \tau/p$ ,

$$\psi_1(Y_1, Y_2) = q - (1/2)p(Y_1 + Y_2) - (1/2)\tau \quad (68)$$

(2) If  $Y_2 + \tau/p \geq Y_1 \geq Y_2 - \tau/p$ ,

$$\psi_1(Y_1, Y_2) = q - pY_1 \quad (69)$$

(3) If  $Y_1 < Y_2 - \tau/p$ ,

$$\psi_1(Y_1, Y_2) = q - (1/2)p(Y_1 + Y_2) + (1/2)\tau \quad (70)$$

A symmetrical solution holds for  $\psi_2$ .

For region 1, the first term in equation (66) becomes:

$$\alpha q - (1/2)\alpha p(C_1 + C_2 + 2\mu) +$$

$$(1/2)\alpha p(C_2 - C_1 - \tau/p) \int_{x_2=0}^{\infty} \int_{x_1=C_1-C_1-\tau/p+x_2}^{C_1-C_1+\tau/p+x_2} f(x_1)dx_1 f(x_2)dx_2 +$$

$$(1/2)\alpha p \int_{x_2=0}^{\infty} \int_{x_1=C_1-C_1-\tau/p+x_2}^{C_2-C_1+\tau/p+x_2} (x_1 + x_2)f(x_1)dx_1 f(x_2)dx_2$$

and the second term becomes one of the following:

(1)  $q - (1/2)p(S_1 + S_2 - C_1 - C_2) - (1/2)\tau$ , if  $S_1 - C_1 > S_2 - C_2 + \tau/p$

(2)  $q - p(S_1 - C_1)$ , if  $S_2 - C_2 + \tau/p \geq S_1 - C_1 \geq S_2 - C_2 - \tau/p$

(3)  $q - (1/2)p(S_1 + S_2 - C_1 - C_2) + (1/2)\tau$ , if  $S_1 - C_1 < S_2 - C_2 - \tau/p$

Symmetrical expressions appear for the equation applying to region 2. The solution for  $n=2$  consists of finding values  $(C_1, C_2)$  to satisfy the two equations for each possible set of values  $(S_1, S_2)$ .

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## APPENDIX

## NOTE 1.—MAJOR SYMBOLS USED

$\alpha$	Discount factor = $1/(1 + \text{interest rate})$
$a$	Constant term in the linear marginal value function
$b$	Absolute value of the slope coefficient in the linear marginal value function
$\beta(S)$	Maximized sum of discounted expected gains in all future years;

$$\beta(S) = \lim_{n \rightarrow \infty} \hat{V}_{1,n}(S)$$

$\beta$	A probability (see page 57)
$\gamma$	Cost of storage function, dollars
$\gamma'$	Marginal cost of storage function per bushel, dollars
$\delta$	Total value function (defined on pages 13–15), dollars
$\rho$	Marginal value, or price, function per bushel, dollars
$\rho_0$	Marginal value per bushel when utilization equals 29.46 bushels, dollars
$\rho_w$	Marginal value function under conditions of war, dollars
$\epsilon$	Flexibility of marginal value function;

$$\epsilon(Y) = -[d\rho(Y)/dY] \cdot [Y/\rho(Y)]$$

$\eta$	Elasticity of marginal value function; $\eta = -1/\epsilon$
$\epsilon_0, \eta_0$	Values of $\epsilon$ and $\eta$ , respectively, at the point where quantity utilized equals 30 bushels per acre
$\mu$	Mean of probability distribution
$\sigma$	Standard deviation of probability distribution
$\theta$	Carryover rule
$\hat{\theta}$	Optimal carryover rule
$\hat{\theta}_i$	This has two meanings, depending on the context: (1) Optimal storage rule in the $i^{\text{th}}$ year; (2) Optimal stationary storage rule under the $i^{\text{th}}$ set of conditions
$\theta_i$	The result obtained at the $i^{\text{th}}$ iteration, in computations to obtain an optimal stationary storage rule
$\theta_w$	Carryover rule under war conditions
$\theta_d$	Carryover rule under conditions of war preparedness
$\hat{\theta}^0$	Optimal carryover rule when harvest in each future year is assumed equal to a known constant
$\lambda(S)$	A function defined on page 56
$\lambda(Z_1, \dots, Z_m)$	A function defined on page 61
$\phi^{-1}$	The function which is inverse to the function $\phi$
$C$	Carryover, bushels
$D$	An operator (see page 54)
$C^*$	Equilibrium level of carryover (defined on pages 56–57), bushels

- C\*\* Level that the carryover will reach after two successive "bumper-crop" years (see page 29), bushels  
 E Mathematical expectation; if  $x_1, \dots, x_n$  are random variables,

$$E\phi(a_1, \dots, a_r, x_1, \dots, x_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi(a_1, \dots, a_r, x_1, \dots, x_n) dF_1(x_1) \dots dF_n(x_n)$$

- $E_i$  Expectation with respect to the random variable  $i$   
 F Probability distribution (usually of  $x$ )  
 f Probability density function of F, or relative frequency  
 G Alternative distribution of output  
 g Probability density function of G  
 J An operator  
 k This has two meanings depending on the context:  
 (1) Value of S below which  $\theta(S)=0$ ;  
 (2) Used occasionally to designate an arbitrary constant  
 L A function (defined on page 55) which depends on the probability distribution of outputs  
 M or K A constant (defined on page 55) whose value depends on the conditions of an application  
 $M_h$  Value of the marginal value function at  $Y=h$   
 m (1) Number of iterations (page 47);  
 (2) Number of regions (page 60)  
 n Number of years and/or number of iterations  
 $P_o$  Some specific value of the marginal value function  $\rho(Y)$   
 $P_t$  Price in year  $t$  (see page 15), dollars  
 $p, q$  Parameters in the linear marginal value function  $\rho(Y)=q-pY$   
 $R_n(S)$  Expected returns to storage in years  $1, \dots, n$  (defined on page 55)  
 r (1) Interest rate (page 11);  
 (2) Also used occasionally to designate an arbitrary constant factor  
 S Total supply in given year=carryover from preceding year plus harvest, bushels

$$S_t = C_{t-1} + X_t$$

- $V_{m, n}$  Sum of discounted expected gains in years  $m, m+1, \dots, n$   
 $\hat{V}_{m, n}$  Maximized  $V_{m, n}$   
 $W_t$  Gain occurring in year  $t$   
 $X$  Harvest or output, bushels  
 Y Quantity utilized, bushels

$$Y = S - C = S - \theta(S)$$

In general, Latin letters that represent quantities are shown in lower case when the quantities are assumed to be random and are

shown as capitalized when the quantities are assumed to be given or determinable.

## NOTE 2.—THE ROSENBLATT SOLUTION

Rosenblatt (8) addresses himself to essentially the same problem as that discussed on page 20, namely, finding a storage rule which, under stationarity, maximizes the sum of expected gains in all future years, where the gain in any year is the total value of the grain utilized minus the cost of storage of the grain carried over.<sup>28</sup> However, for mathematical convenience, he restricts himself to:

(1) A form of storage rule which makes the carryover in any year a constant proportion (to be determined) of the total available supply (carryin plus harvest), and

(2) Application of the criterion of optimality *only after* the probability distributions of carryovers and quantities consumed ( $C$  and  $Y$ ) have completely stabilized. This restriction means that, in any practical application of the rules, their effects during the first several years of operation are completely ignored.

The combined effects of these two restrictions or assumptions lead to storage rules which are in fact highly nonoptimal under the criterion adopted, and which, if taken seriously as guides to public policy, would result in the incurring of costs to the nation as a whole possibly running into hundreds of millions of dollars.

The objections to the Rosenblatt approach may be outlined in greater detail as follows:

1. It is *not* necessary to make in advance any assumption about the form of the storage rule. The method of solution presented by Dvoretzky, Kiefer, and Wolfowitz (2) (as modified in this bulletin) permits the obtaining of solutions without any such prior assumption.

2. Optimal storage rules under the conditions and the criterion adopted here (and the criterion of Section 3 of Rosenblatt's paper) do not in fact turn out to have anything like the form assumed by Rosenblatt (see p. 69).

3. It can be shown that, using empirically plausible assumptions about the other conditions, a constant-proportion storage rule *cannot* be optimal, unless the cost of storage function is assumed to take a form which is empirically highly implausible. Consider equation (26.1) shown on page 46. With  $\theta(S) = aS$ , this gives us

$$\gamma'(C) = \alpha E\rho[(1-a)(C+x)] - \rho[(1-a)C/a] \quad (71)$$

$$\gamma'(0) = \alpha E\rho[(1-a)x] - \rho(0) \quad (72)$$

That is, the marginal cost of storage at  $C=0$  is highly negative. For example, if  $\rho$  is linear (corresponding to Rosenblatt's use of a quadratic weight function) and  $\rho(Y) = q - pY$ , then

$$\gamma'(0) = -(1-\alpha)q - p(1-\alpha)\mu \quad (72.1)$$

where  $\mu$  is the mean yield.

Also,

$$\gamma''(C) = \alpha(1-a)E\rho'[(1-a)(C+x)] - (1-a)\rho'[(1-a)C/a]/a \quad (73)$$

<sup>28</sup> Actually, Rosenblatt's criterion is stated as the minimization of the sum of expected "losses" in all future years, where the loss in any year is the "weighting" attributable to the quantity of grain utilized plus the cost of storage of the grain carried over. But the weighting function is simply the negative of our total value function, plus a constant; so that the two criteria are mathematically equivalent.

If  $\rho(Y) = q - pY$ , then  $\rho'(Y) = -p$ , and

$$\gamma''(C) = -\alpha(1-a)p + (1-a)p/a = (1-\alpha)(1-a)p/a \quad (73.1)$$

Thus for  $\rho(Y) = q - pY$ ,  $\gamma'(C) = 0$  when

$$C = aq/(1-a)p + a\mu/(1-\alpha) \quad (74)$$

We can minimize the non-optimality of the Rosenblatt results if, instead of taking the carryover as a certain proportion of total supply ( $S$ ), we make it a certain proportion of total supply minus the minimum possible harvest ( $S - x_{min}$ ). This does not change any of the mathematics of the solution, but means simply that we are "changing the origin" in the measurements of  $S$ ,  $Y$  and  $X$ . This modification, which minimizes the degree to which constant-proportion rules deviate from optimality and hence presents the Rosenblatt results in their most favorable light, is used in the following comments where we compare constant-proportion rules with optimal rules.

Applying the above results to a specific case, for example, to the conditions applicable to  $\hat{\theta}_6$  (see page 30), we find that a constant-proportion rule is optimal *only* if the marginal cost of storage  $\gamma'(C)$  is *negative* up to a carryover  $C$  of about 18 bushels per acre, which is more than three times the average carryover that results under the Rosenblatt solution.

4. The Rosenblatt results maximize<sup>29</sup> the sum of discounted expected gains, starting with the current year, *only* if the current initial supply,  $S$ , equals the long-run expected stable or ergodic value of  $S$ . If the initial  $S$  is any other value, the gains and costs of the storage program during the first several years of application of the rule, before stability in all the probability distributions is attained, are simply ignored. But in the sum of discounted expected gains, the first years of the period are the most important, and a storage policy, to be practicable, should be applicable to *any* set of initial conditions. One result of Rosenblatt's restriction is that nowhere in his solution does a discount factor or interest rate appear; this alone would indicate that the validity of the solution is, from an economic viewpoint, rather implausible.

5. As a result of his assumptions, the Rosenblatt storage rules bear little resemblance to rules which are in fact optimal. They do not even result in a correct order of magnitude of carryover levels, under alternative sets of conditions. Consider the seven alternative sets of conditions underlying optimal rules  $\hat{\theta}_1, \dots, \hat{\theta}_7$ , respectively, as shown in table 1. The storage-rule proportion,  $a$ , which minimizes expected losses, and the resulting stable expected value of carryover,  $EEC$ , under the Rosenblatt solution, are given by:

$$a = \frac{\sqrt{p\sigma^2/\gamma'\mu} - 1}{\sqrt{p\sigma^2/\gamma'\mu} + 1} \quad (75)$$

$$EEC = a\mu/(1-a) \quad (76)$$

where  $\sigma^2$  = variance of yields,  $\mu$  = mean yield, and  $\gamma'$  = (constant) marginal cost of storage. If  $p\sigma^2/\mu\gamma' < 1$ ,  $a = 0$ .

Values of  $a$  and  $EEC$  for the seven sets of conditions, taking the origin for  $S$ ,  $Y$  and  $X$  at  $x_{min} = 19$  bushels per acre, are shown in table 6, together with the equilibrium level,  $C^*$ , of the corresponding optimal rule.

A graphical comparison of rules that result from the Rosenblatt approach and the optimal rules developed in this bulletin is shown in figure 8, using the same alternative conditions as in table 6.

6. An idea of the magnitude of the economic loss to society that would be incurred by adopting the Rosenblatt solution, instead of using optimal storage rules, is obtained by using the concept of expected returns to storage, as defined on page 55: the difference between the sum of discounted expected gains when the optimal policy is followed and the sum of discounted expected gains when the carryover in every year is zero. We may readily extend this concept so as to apply it to any storage policy, whether optimal or nonoptimal: for any given storage policy, the expected return is the difference between the sum of discounted

<sup>29</sup> Subject to his constant-proportion storage rule restriction.

TABLE 6.—*Corn, oats, and barley, corn equivalent: Storage rule proportion and resulting expected stable carryover per acre under the Rosenblatt solution compared with the equilibrium carryover level per acre under an optimal rule*

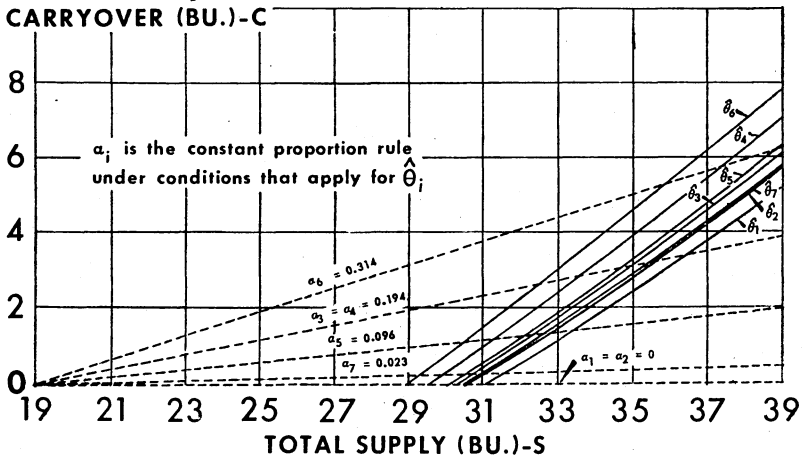
Case <sup>1</sup>	Rosenblatt results		Equilibrium carryover (C*)
	Storage rule proportion (a)	Average carry-over (EEC)	
		<i>Bushels</i>	<i>Bushels</i>
1.....	0	0	0.3
2.....	0	0	.5
3.....	.19	2.5	.6
4.....	.19	2.5	1.4
5.....	.10	1.1	.7
6.....	.31	4.8	2.7
7.....	.02	.2	.4

<sup>1</sup> See table 1 for specified conditions.

expected gains when the given policy is followed and the sum of discounted expected gains when the carryover in every year is zero. The expected social loss, then, incurred by following any given nonoptimal policy may be defined as the expected return to the *optimal* policy computed for the given conditions *minus* the expected return to the given nonoptimal policy.

## FEED GRAINS\*: STORAGE RULES PER ACRE OPTIMAL RULES COMPARED WITH CONSTANT PROPORTION RULES

Under Alternative Conditions Specified in Table 1



\*CORN, OATS AND BARLEY, CORN EQUIVALENT.

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FIGURE 8.—Storage rules developed under the constant-proportion assumption used by Rosenblatt differ greatly from the optimal rules developed in this bulletin and, if taken seriously as guides to public policy, would result in large costs to the nation as a whole.

The expected returns for the optimal rule  $\hat{\theta}_1$  (see table 1) have been computed for alternative values of the initial supply  $S$ , and are given in the following tabulation:

Per acre	
Initial supply	Expected return
<i>Bushels</i>	<i>Dollars</i>
31 or less.....	0. 32
32.....	. 35
34.....	. 58
36.....	. 95
38.....	1. 80
40.....	2. 83
42.....	4. 16
44.....	5. 79
46.....	7. 67
48.....	9. 52
50.....	11. 28

The Rosenblatt solution, when applied to the conditions for which  $\hat{\theta}_1$  is optimal, results in zero carryover in every year (see table 6). The expected return under this policy would therefore be zero for any initial supply  $S$ . Hence the expected returns for the optimal policy are in this case equal to the expected losses that would be incurred under the Rosenblatt solution policy. When we multiply the above figures by 140 million acres to convert them to national aggregates, the expected losses range from a minimum of about \$45 million to a possible maximum of \$1,500 million or higher, depending on the level of initial supply.

As another example, consider the conditions for which  $\hat{\theta}_6$  (table 1) is optimal. The expected returns under  $\hat{\theta}_6$ , for alternative levels of initial supply  $S$ , are:

Per acre	
Initial supply	Expected return
<i>Bushels</i>	<i>Dollars</i>
29 or less.....	11. 41
30.....	11. 48
32.....	11. 97
34.....	12. 97
36.....	14. 50
38.....	16. 57
40.....	19. 08
42.....	21. 58
44.....	23. 98
46.....	26. 30
48.....	28. 54
50.....	30. 71

The Rosenblatt solution, when applied to the conditions for which  $\hat{\theta}_0$  is optimal, and taking the S-axis intercept of the storage rule at  $x_{min}=19$  bushels per acre (to minimize the non-optimality of the solution), results in a storage rule proportion  $a=0.314$  and a long-run expected carryover  $EEC=4.78$  bushels per acre (see table 6). We have not computed a complete table of expected returns under this rule, for alternative values of initial supply  $S$ . However, a comparison can be made between the two rules by taking the situation *most favorable* to the Rosenblatt rule, namely at the point where the initial supply  $S=34.24$  bushels per acre, the long run expected level of  $S$  which corresponds to the long run expected carryover  $EEC=4.78$ . For an initial  $S=34.24$ , the expected return under the *optimal* rule is, by interpolation in the above table, about \$13.12 per acre. The expected return under the constant-proportion rule (for initial  $S=34.24$ ) is obtained as follows:

- (1) Net gain in current year  
 $= (\text{Total value under rule}) - (\text{Cost of storage})$   
 $\quad \quad \quad - (\text{Total value with zero carryover}) \quad (77)$

$$= (q\mu - p\mu^2/2) - (\gamma' EEC) - [34.24 q - p (34.24)^2/2] \quad (77.1)$$

since  $\mu$  is the amount utilized under the rule and 34.24 is the amount utilized with zero carryover. Substituting  $q=6.50$ ,  $p=0.167$  (p. 24),  $\gamma'=0.04$  (table 1),  $EEC=4.78$ , and  $\mu=29.46$ , gives:

$$\text{Net gain in current year} = -\$5.92. \quad (77.2)$$

(The negative sign, indicating a net loss, is, of course, what we should expect.)

- (2) Expected net gain in each future year  
 $= E (\text{Total value under rule}) - E (\text{Cost of storage})$   
 $\quad \quad \quad - E (\text{Total value with zero carryover}) \quad (78)$

$$= EE(qY - pY^2/2) - \gamma' EEC - E(qX - pX^2/2) \quad (78.1)$$

$$= p(\text{Var } X - \text{Var } Y)/2 - \gamma' EEC \quad (78.2)$$

where  $\text{Var } X$  is the variance of  $X$  and  $\text{Var } Y$  is the long run (stable) variance of  $Y$ . (The last step makes use of the fact that the long run expected value of  $Y$ ,  $EEY=\mu$ .) It can be shown that  $\text{Var } Y = (1-a)^2\sigma^2/(1-a^2)$ , where  $\sigma^2 = \text{Var } X$ , so that  $\text{Var } X - \text{Var } Y = 2a\sigma^2/(1+a)$ . Substituting  $p=0.167$ ,  $\sigma^2=9.18$ ,  $a=0.314$ ,  $\gamma'=0.04$ , gives:

$$\text{Expected net gain in each future year} = \$0.175. \quad (78.3)$$

- (3) The sum of discounted expected net gains in all future years is obtained by multiplying the result of (2) by  $\sum_{n=1}^{\infty} \alpha^n = \alpha/(1-\alpha)$ . Substituting  $\alpha=0.98$  (table 1) gives:

Sum of discounted expected net gains in all future years=  
 $(\$0.175) (49) = \$8.59.$  (79)

(4) Adding the results of (1) and (3) gives:

Expected return under the constant-proportion rule (for initial  
 $S = 34.24$ ) equals  $\$8.59 - \$5.92 = \$2.67.$  (80)

Comparing the expected return of \$2.67 under the constant-proportion rule with the expected return of \$13.12 under the optimal rule for the same conditions, we have an expected loss to the entire nation of  $\$13.12 - \$2.67 = \$10.45$  per acre incurred by adopting the Rosenblatt solution instead of an optimal rule, even under the assumption about initial supply which is most favorable to the former. Multiplying the per acre loss by 140 million acres gives a national aggregate loss of about \$1,500 million.

### NOTE 3.—A STORAGE RULE UNDER WHICH THE ADDITION TO CARRYOVER IS A FUNCTION OF CURRENT CROP ONLY

A storage program might be thought of as an attempt to decrease the variance of a probability distribution, that is, an attempt to convert the distribution of outputs into a distribution of quantities utilized with the same shape but smaller variance. The objections, operational and analytical, to the direct application of such a concept in the derivation of storage rules were set forth on page 8. A numerical example illustrating the details of how such a direct application would work out is given here.

Let us consider wheat alone, and treat the United States as a single, closed market. We assume a constant acreage of 68 million acres, approximately the average for 1919–50. Actual yields per seeded acre for all wheat during 1919–50 are considered as random independent observations. We thus have a sample of 32 observations with a mean of 13.05 bushels per acre and a standard deviation of 2.60, and an approximately normal distribution.<sup>30</sup> We assume, then, that annual output ( $X$ ) is normally distributed with mean  $\mu_x = 13.05 \times 68 = 887$  million bushels and standard deviation  $\sigma_x = 2.60 \times 68 = 177$  million bushels. The probability of output falling more than 20 percent below average is about 16 percent.

Suppose we wish to make the amount added to carryover a function of the current crop and to alter the variance of the distribution so that the probability that the quantity of wheat utilized ( $Y$ ) in any year will fall more than 20 percent below average is reduced to 5 percent, instead of 16 percent. The criterion must be kept in terms of probabilities, unless we go to the extreme of complete stabilization, or unless we state the criterion in terms of the change in the variance itself. The simplest form for such a rule is  $Z = 0.39 (X - 887)$ , where  $Z$  is the amount to be added to carryover.  $Z$  can be positive or nega-

<sup>30</sup> A goodness-of-fit test for normality gives a probability level for  $\chi^2$  of more than 90 percent.

tive, of course. This rule would result, under the assumption stated, in a normal probability distribution of  $Z$  with mean  $\mu_z=0$  and standard deviation  $\sigma_z=69$  and a normal probability distribution of quantity utilized with  $\mu_y=887$  and  $\sigma_y=108$ . Here,  $Y=X-Z$ , that is, the loss of grain in the storage operation itself is assumed negligible.

Since we assume independence in yields between years, the total amount added to storage after  $n$  years of operating the rule is

$$Z_1+Z_2+\dots+Z_n=\Sigma Z \quad (81)$$

a normally distributed variable with zero mean and standard deviation  $\sigma=\sqrt{n} \cdot \sigma_z=69\sqrt{n}$ . Since the first years of the period of application of the rule may themselves be years of low yields, it would be necessary to start the period with grain on hand. Thus, for example, in order to be 99 percent sure of having enough grain in storage to operate the rule for one year, the storing agency would need to start the year with 161 million bushels on hand. To be 99 percent sure of having enough grain in storage to operate the rule for 9 years, the agency would have to start the period with 483 million bushels on hand.

The effects of allowing for sampling error in the distribution estimates also can be illustrated. Confidence interval estimates at the 90 percent probability level for the mean and standard deviation of yield are:  $12.27<\mu<13.83$  and  $2.16<\sigma<3.30$ . Based on the national aggregate of 68 million acres, the confidence intervals for output are:  $835<\mu_x<941$  and  $147<\sigma_x<224$ . If we (1) ignore the possible error in the mean<sup>31</sup> but take each of the confidence limits for  $\sigma_x$  and (2) use the same criterion for stability in quantities utilized and the same kind of storage rule as previously, results shown in table 7 are obtained.

TABLE 7.—Wheat: Upper and lower limits for storage rules and related quantities obtained when the addition to carryover is a function of current production only and allowance is made for sampling error in the standard error of output

Item	Unit	Limit	
		Lower	Upper
Standard deviation:			
Output, $\sigma_x$ <sup>1</sup> -----	Mil. bu. --	147	224
Storage rule, $\sigma_x$ -----	do.-----	39	116
Probability of an output less than 80 percent of average.	Pct.-----	11	22
Storage rule as a proportion of the deviation of the crop from average, $Z$ .	-----	.26	.52
Initial stocks required to be able to operate the rule with 99 percent certainty for—			
1 year-----	Mil. bu. --	91	270
9 years-----	do.-----	273	810

<sup>1</sup> Limits shown are based on a confidence interval at the 90 percent probability level.

<sup>31</sup> Errors of this sort imply that application of the rule results in the level of carryover trending upward without bound, or downward to zero.

**NOTE 4.—RELATIVE IMPORTANCE ON OUTPUT OF VARIATIONS IN ACREAGE AND YIELD**

The total variation in acreage normally is quantitatively of less importance than the variation in yield in its effect on variability of output. This is illustrated by the data for wheat and corn in the United States as shown in table 8.

TABLE 8.—*Wheat and corn: Relative variability in acreage and yield per seeded acre as indicated by specified coefficients*<sup>1</sup>

Item	Wheat, 1919–50		Corn, 1929–50	
	Acres	Yield	Acres	Yield
	<i>Millions</i>	<i>Bushels</i>	<i>Millions</i>	<i>Bushels</i>
Mean.....	68.2	13.0	95.5	28.3
Standard deviation.....	7.4	2.6	8.4	7.2
Range.....	31.7	10.3	28.6	28.0
Minimum.....	53.0	8.0	84.4	14.4
Maximum.....	84.7	18.3	113.0	42.4
Average year-to-year change.....	4.6	1.4	3.1	4.0
As a percentage of the mean:	<i>Percent</i>	<i>Percent</i>	<i>Percent</i>	<i>Percent</i>
Standard deviation.....	10.8	19.9	8.8	25.4
Range.....	46.5	78.9	29.9	98.9
Minimum minus the mean.....	–22.3	–38.7	–11.6	–49.1
Maximum minus the mean.....	24.2	40.2	18.3	49.8
Average change.....	6.8	10.8	3.3	14.1

<sup>1</sup> Published series on yield per seeded acre begin in 1919 for wheat and in 1929 for corn.

The total variation in acreage is made up of predictable changes as well as unpredictable. If we were to compare the relative magnitudes of *unpredictable* variations in acreage and yield, the former would be of still less importance than indicated by the figures for total variation.

This subject is discussed in detail in a recent Senate Committee report (12, pp. 17–30).

**NOTE 5.—THE SOLUTION USING THE TOTAL VALUE ( $\delta$ ) FUNCTION**

As has been indicated, this solution is adapted from that by Dvoretzky, Kiefer, and Wolfowitz (2). Some modification was required because of the different structure of the problem. Also, the concepts “returns to storage” and “equilibrium level of carryover” do not have counterparts in the inventory problem considered by these authors.

The solution as written out here assumes, for simplicity's sake, independence in probability distributions of yields between years. Modifications required to incorporate joint probability distributions of yields in all years are not formally serious, though they would in general substantially increase the number of computations. Modifications required for serial dependence of specified kinds are discussed on p. 59.

We first give the solution for a set of optimal storage rules for an  $n$ -year period, with no assumption of stationarity. Using the notation in the body of the bulletin, we generalize by letting  $V_{m,n}$  be the sum of expected gains in years  $m, m+1, m+2, \dots, n$ , discounted back to year  $m$ . That is,

$$V_{m,n} = W_m + \alpha E W_{m+1} + \alpha^2 E W_{m+2} + \dots + \alpha^{n-m} E W_n \quad (82)$$

If  $\hat{V}_{m,n}$  is the maximum  $V_{m,n}$  for given  $S_m$ , we have:

$$\hat{V}_{n,n} = \text{Max}_{0 \leq C_n \leq S_n} [\delta_n (S_n - C_n) - \gamma_n (C_n)] = \delta_n (S_n) \quad (83.1)$$

$$\hat{V}_{n-1,n} = \text{Max}_{0 \leq C_{n-1} \leq S_{n-1}} \left[ \delta_{n-1} (S_{n-1} - C_{n-1}) - \gamma_{n-1} (C_{n-1}) + \alpha \int_0^\infty \delta_n (C_{n-1} + x) dF_n(x) \right] \quad (83.2)$$

$$= \psi_{n-1} (S_{n-1}) \quad (\text{say}) \quad (84.2)$$

$$\hat{V}_{n-2,n} = \text{Max}_{0 \leq C_{n-2} \leq S_{n-2}} \left[ \delta_{n-2} (S_{n-2} - C_{n-2}) - \gamma_{n-2} (C_{n-2}) + \alpha \int_0^\infty \psi_{n-1} (C_{n-2} + x) dF_{n-1}(x) \right] \quad (83.3)$$

$$= \psi_{n-2} (S_{n-2}) \quad (\text{say}) \quad (84.3)$$

and so on, till we reach

$$\hat{V}_{1,n} = \text{Max}_{0 \leq C_1 \leq S_1} \left[ \delta_1 (S_1 - C_1) - \gamma_1 (C_1) + \alpha \int_0^\infty \psi_2 (C_1 + x) dF_2(x) \right] \quad (83.4)$$

$$= \psi_1 (S_1) \quad (\text{say}) \quad (84.4)$$

$\hat{V}_{m,n}$  is thus a function of  $S_m$  obtained by maximizing, for each value of  $S_m$ , the expression in square brackets. The optimum carryover for year  $m$ , for given  $S_m$ , is that value  $\hat{C}_m$  which maximizes the same expression. The optimal storage rule  $\hat{\theta}_m$  is the set of all such pairs  $(S_m, \hat{C}_m)$ .

Thus, the computations are actually carried out on the gain functions, with the storage rules coming out more or less as by-products. This is the complete solution for the  $n$ -year non-stationary case.

For the stationary case, where  $\delta, \gamma$  and  $F$  are the same in each year, we note first that  $\hat{V}_{1, 1+m}$  as a function of  $S_1$  is the same as  $\hat{V}_{n, n+m}$  as a function of  $S_n$  for any  $m$  and  $n$ . We now define the operator  $J$ , operating on any function  $\phi$ , by

$$J\phi(S) = \text{Max}_{0 \leq C \leq S} \left[ \delta(S - C) - \gamma(C) + \alpha \int_0^\infty \phi(C + x) dF(x) \right] \quad (85)$$

Then, omitting subscripts from  $S_1$  and  $C_1$ :

$$\hat{V}_{1,1} = \text{Max}_{0 \leq C \leq S} [\delta(S-C) - \gamma(C)] = \delta(S) \quad (85.0)$$

$$\hat{V}_{1,2} = \text{Max}_{0 \leq C \leq S} \left[ \delta(S-C) - \gamma(C) + \alpha \int_0^\infty \delta(C+x) dF(x) \right] = J\delta(S) \quad (85.1)$$

$$\hat{V}_{1,3} = \text{Max}_{0 \leq C \leq S} \left[ \delta(S-C) - \gamma(C) + \alpha \int_0^\infty J\delta(C+x) dF(x) \right] = J^2\delta(S) \quad (85.2)$$

and in general

$$\hat{V}_{1,n} = \text{Max}_{0 \leq C \leq S} \left[ \delta(S-C) - \gamma(C) + \alpha \int_0^\infty J^{n-2}\delta(C+x) dF(x) \right] = J^{n-1}\delta(S) \quad (85.3)$$

For given  $n$ , we obtain  $\hat{\theta}_1$ , the optimal rule for the first year, by noting for each value of  $S$  the value of  $C$  that maximizes the expression in square brackets.

The only remaining question is, Does the process converge so that as  $n$  gets larger and larger the resulting  $\hat{\theta}_1$  gets closer and closer to the best stationary rule  $\hat{\theta}$ ? In other words, if we designate by  $\beta(S)$  the sum of discounted expected gains to infinity when the best rule  $\hat{\theta}$  is followed every year, does  $\lim_{n \rightarrow \infty} J^n \delta(S) = \beta(S)$ ? This seems obvious,

but in any case, a formal proof of a stronger statement can be offered, namely, that if  $g(S)$  is *any* bounded function, then  $\lim_{n \rightarrow \infty} J^n g(S) = \beta(S)$ .

This implies that we could reduce the number of iterations necessary to achieve a given closeness of approximation to  $\hat{\theta}_1$  by starting with some  $g(S)$  which is closer to  $\beta(S)$  than is  $\delta(S)$ . This result was not used in the actual computations, however, because (1) it was not obvious how to find a  $g(S)$  that would be much better to start with than  $\delta(S)$  itself; (2) it was felt to be somewhat advantageous to follow a procedure with as much intuitive plausibility as possible; (3) by starting with  $\delta$ , each iteration produces in itself a result which has common sense meaningfulness, that is, a storage rule which is the optimal rule for the first year of an  $n$ -year period (in the case of the  $(n-1)^{\text{th}}$  iteration).

However, the proof, which like the rest of this discussion is adapted from Dvoretzky, Kiefer, and Wolfowitz (2), is recorded here for possible use in future applications. We first break the operator  $J$  into two parts  $I$  and  $G$  so that  $J = I G$  where

$$I\phi(S) = \sup_{0 \leq C \leq S} [\delta(S-C) + \phi(C)] \quad (86)$$

$$G\phi(C) = -\gamma(C) + \alpha \int_0^\infty \phi(C+x) dF(x) \quad (87)$$

With  $\beta$  defined as in the preceding paragraph, the maximized sum of discounted expected gains in all intervals except the first, is

$$\alpha \int_0^{\infty} \beta(C_1+x) dF(x) \quad (88)$$

So we can write:

$$\begin{aligned} \beta(S) &= \sup_{0 \leq C \leq S} [\delta(S-C) - \gamma(C) + \alpha \int_0^{\infty} \beta(C+x) dF(x)] \\ &= J\beta(S) = IG\beta(S) \end{aligned} \quad (89)$$

Since  $\delta(S)$  is bounded,  $\beta(S)$  also must be bounded and so must  $\beta(S) - g(S)$ . Let

$$\sup_{S \geq 0} |\beta(S) - g(S)| = M \quad (90)$$

Then

$$\sup_{S \geq 0} |G\beta(S) - Gg(S)| \leq M \quad (91)$$

since

$$G\beta(S) - Gg(S) = \alpha \int_0^{\infty} [\beta(S+x) - g(S+x)] dF(x) \quad (91.1)$$

$$\leq \alpha \int_0^{\infty} \sup_{R \geq 0} |\beta(R) - g(R)| dF(x) \quad (91.2)$$

$$= \alpha \sup_{R \geq 0} |\beta(R) - g(R)| = \alpha M \quad (91.3)$$

Also,

$$\sup_{S \geq 0} |IG\beta(S) - IGg(S)| \leq \alpha M \quad (92)$$

To prove this, we must show that

$$\sup_{S \geq 0} |I\phi_1(S) - I\phi_2(S)| \leq \sup_{S \geq 0} |\phi_1(S) - \phi_2(S)| \quad (92.1)$$

For given  $S$ , let

$$\delta(S-R) + \phi_1(R) = \psi_1(R) \quad (0 \leq R \leq S) \quad (92.3)$$

and

$$\delta(S-Z) + \phi_2(Z) = \psi_2(Z) \quad (0 \leq Z \leq S) \quad (92.4)$$

Then

$$\inf_{0 \leq R \leq S} \psi_1(R) - \psi_2(Z) \leq \psi_1(R) - \psi_2(Z) \quad \begin{cases} 0 \leq R \leq S \\ 0 \leq Z \leq S \end{cases} \quad (92.5)$$

and

$$\inf_{0 \leq R \leq S} \psi_1(R) - \psi_2(Z) \leq \psi_1(Z) - \psi_2(Z) \quad \begin{cases} 0 \leq R \leq S \\ 0 \leq Z \leq S \end{cases} \quad (92.6)$$

We take sup with respect to  $Z$  on both sides:

$$\inf_{0 \leq R \leq S} \psi_1(R) - \inf_{0 \leq Z \leq S} \psi_2(Z) \leq \sup_{0 \leq Z \leq S} [\psi_1(Z) - \psi_2(Z)] \quad (92.7)$$

$$I\phi_1(S) - I\phi_2(S) \leq \sup_{0 \leq Z \leq S} [\phi_1(Z) - \phi_2(Z)] \quad (92.8)$$

Now let  $S$  vary, and take sup with respect to  $S$ :

$$\sup_{S \geq 0} [I\phi_1(S) - I\phi_2(S)] \leq \sup_{S \geq 0} [\phi_1(S) - \phi_2(S)] \quad (92.9)$$

Similarly, by switching subscripts:

$$\sup_{S \geq 0} [I\phi_2(S) - I\phi_1(S)] \leq \sup_{S \geq 0} [\phi_2(S) - \phi_1(S)] \quad (92.10)$$

Therefore,

$$\sup_{S \geq 0} |I\phi_1(S) - I\phi_2(S)| \leq \sup_{S \geq 0} |\phi_1(S) - \phi_2(S)| \quad (92.11)$$

So we have shown that

$$\sup_{S \geq 0} |J\beta(S) - Jg(S)| \leq \alpha M \quad (92)$$

Repeating  $n$  times, we have

$$\sup_{S \geq 0} |J^n \beta(S) - J^n g(S)| \leq \alpha^n M \quad (93)$$

and since  $J\beta(S) = \beta(S)$ , we conclude

$$\lim_{n \rightarrow \infty} J^n g(S) = \beta(S) \quad (94)$$

This completes the proof.

We have, then, the result that  $J^n \delta(S)$  approaches a limit as  $n$  gets larger, and the resulting  $\hat{\delta}_1$  converges to the best stationary rule  $\hat{\delta}$ . The question arises, How close are we to convergence after any given number of iterations? This cannot be answered exactly, of course (if it could, we would be through before we started), but the speed of convergence can be seen by taking the difference  $J^n \delta(S) - J^{n-1} \delta(S)$ . In the limit, this must be zero, and one can continue the iterations till it is as close to zero as desired. In practice, however, it turns out that this difference becomes nearly a constant long before it diminishes to zero. It can easily be shown that if  $J^n \delta(S) - J^{n-1} \delta(S)$  were a constant, then the storage rule would have reached convergence, as further iterations would make no further change. Hence in most cases little is gained by continuing the iterations beyond the point where  $J^n \delta(S) - J^{n-1} \delta(S)$  is nearly constant.

**NOTE 6.—MATHEMATICAL EQUIVALENCE OF SOLUTION PROCEDURES USING THE MARGINAL VALUE ( $\rho$ ) FUNCTION AND THE TOTAL VALUE ( $\delta$ ) FUNCTION**

*Year n.*— $C=0$  under each procedure.

*Year n-1.*—Using  $\delta$ :

$$V_{n-1,n} = \delta(S-C) - \gamma(C) + \alpha E\delta(C+x) \quad (95.1)$$

To maximize with respect to  $C$ :

$$dV_{n-1,n}/dC = -\rho(S-C) - \gamma'(C) + \alpha E\rho(C+x) = 0 \quad (95.2)$$

This is identical to the condition used in the  $\rho$ -procedure.

We now check the second order condition:

$$d^2V_{n-1,n}/dC^2 = \rho'(S-C) - \gamma''(C) + \alpha E\rho'(C+x) < 0 \quad (95.3)$$

since  $\rho' < 0$  and  $\gamma'' \geq 0$ .

*Year n-2.*—Using  $\delta$ :

$$V_{n-2,n} = \delta(S-C) - \gamma(C) + \alpha E\hat{V}_{n-1,n}(C+x) \quad (96.1)$$

To maximize with respect to  $C$ :

$$dV_{n-2,n}/dC = -\rho(S-C) - \gamma'(C) + \alpha E\hat{V}'_{n-1,n}(C+x) = 0 \quad (96.2)$$

Using  $\rho$ :

$$-\rho(S-C) - \gamma'(C) + \alpha E\rho[C+x - \hat{\theta}_{n-1}(C+x)] = 0 \quad (96.3)$$

For equivalence, we must show that

$$\hat{V}_{n-1,n}(S) = \rho[S - \hat{\theta}_{n-1}(S)] \quad (97)$$

for every  $S$ .

$$\hat{V}_{n-1,n}(S) = \delta[S - \hat{\theta}_{n-1}(S)] - \gamma[\hat{\theta}_{n-1}(S)] + \alpha E\delta[\hat{\theta}_{n-1}(S) + x] \quad (97.1)$$

$$\begin{aligned} \hat{V}'_{n-1,n}(S) &= \rho[S - \hat{\theta}_{n-1}(S)] \cdot [1 - \hat{\theta}'_{n-1}(S)] - \gamma'[\hat{\theta}_{n-1}(S)] \cdot \hat{\theta}'_{n-1}(S) + \\ &\quad \alpha E\rho[\hat{\theta}_{n-1}(S) + x] \cdot \hat{\theta}'_{n-1}(S) \end{aligned} \quad (97.2)$$

$$\begin{aligned} &= \rho[S - \hat{\theta}_{n-1}(S)] + \hat{\theta}'_{n-1}(S) \cdot \{ -\rho[S - \hat{\theta}_{n-1}(S)] - \\ &\quad \gamma'[\hat{\theta}_{n-1}(S)] + \alpha E\rho[\hat{\theta}_{n-1}(S) + x] \} \end{aligned} \quad (97.3)$$

$$= \rho[S - \hat{\theta}_{n-1}(S)]. \quad (97)$$

This completes the proof of equation (97).

The second order condition is:

$$\begin{aligned} d^2V_{n-2,n}/dC^2 &= \rho'(S-C) - \gamma''(C) + \alpha E\rho'[C+x - \hat{\theta}_{n-1}(C+x)] \cdot \\ &\quad [1 - \hat{\theta}'_{n-1}(C+x)] < 0 \end{aligned} \quad (98)$$

since  $0 \leq \theta' < 1$ .

To complete the proof, we show that if equivalence holds for  $n$ ,  $n-1$ , . . . ,  $n-k+1$ ,  $n-k$ , then it holds for  $n-k-1$ :

For  $n-k$ :

$$dV_{n-k,n}/dC = -\rho(S-C) - \gamma'(C) + \alpha E \hat{V}'_{n-k+1,n}(C+x) = 0 \quad (99.1)$$

and

$$-\rho(S-C) - \gamma'(C) + \alpha E \rho[C+x - \hat{\theta}_{n-k+1}(C+x)] = 0 \quad (99.2)$$

Hence,

$$\alpha E \hat{V}'_{n-k+1,n}(C+x) = \alpha E \rho[C+x - \hat{\theta}_{n-k+1}(C+x)] \quad (99.3)$$

and

$$\alpha E \hat{V}'_{n-k+1,n}[\hat{\theta}_{n-k}(S) + x] = \alpha E \rho\{\hat{\theta}_{n-k}(S) + x - \hat{\theta}_{n-k+1}[\hat{\theta}_{n-k}(S) + x]\} \quad (99.4)$$

For  $n-k-1$ :

Using  $\delta$ :

$$V_{n-k-1,n} = \delta(S-C) - \gamma(C) + \alpha E \hat{V}_{n-k,n}(C+x) \quad (100.1)$$

To maximize:

$$dV_{n-k-1,n}/dC = -\rho(S-C) - \gamma'(C) + \alpha E \hat{V}'_{n-k,n}(C+x) = 0 \quad (100.2)$$

Using  $\rho$ :

$$-\rho(S-C) - \gamma'(C) + \alpha E \rho[C+x - \hat{\theta}_{n-k}(C+x)] = 0 \quad (100.3)$$

For equivalence, we must show that

$$\hat{V}'_{n-k,n}(S) = \rho[S - \hat{\theta}_{n-k}(S)] \quad (101)$$

for every  $S$ .

$$\hat{V}_{n-k,n}(S) = \delta[S - \hat{\theta}_{n-k}(S)] - \gamma[\theta_{n-k}(S)] + \alpha E \hat{V}_{n-k+1,n}[\hat{\theta}_{n-k}(S) + x] \quad (101.1)$$

$$\hat{V}'_{n-k,n}(S) = \rho[S - \hat{\theta}_{n-k}(S)] \cdot [1 - \hat{\theta}'_{n-k}(S)] - \gamma'[\hat{\theta}_{n-k}(S)] \cdot \hat{\theta}'_{n-k}(S) + \quad (101.2)$$

$$\alpha E \hat{V}'_{n-k+1,n}[\hat{\theta}_{n-k}(S) + x] \cdot \hat{\theta}'_{n-k}(S)$$

$$= \rho[S - \hat{\theta}_{n-k}(S)] + \hat{\theta}'_{n-k}(S) \{ -\rho[S - \hat{\theta}_{n-k}(S)] -$$

$$\gamma'[\hat{\theta}_{n-k}(S)] + \alpha E \hat{V}'_{n-k+1,n}[\hat{\theta}_{n-k}(S) + x] \} \quad (101.3)$$

$$= \rho[S - \hat{\theta}_{n-k}(S)] \quad (101)$$

This completes the proof of equivalence.

The second order condition is:

$$d^2V_{n-k-1,n}/dC^2 = \rho'(S-C) - \gamma''(C) + \alpha E \rho'[C+x - \hat{\theta}_{n-k}(C+x)]$$

$$[1 - \hat{\theta}'_{n-k}(C+x)] < 0 \quad (102)$$

**NOTE 7.—EQUIVALENCE OF CHANGES IN  $\rho$  AND  $\gamma'$  (SEE PAGE 49)**

(1) If  $\theta$  satisfies

$$\alpha \int_0^{\infty} \rho^*[C+x-\theta(C+x)]dF(x) - \rho^*[\theta^{-1}(C)-C] - \gamma'(C) = 0 \quad (103.1)$$

where  $\rho^*(Y) = r\rho(Y)$ , then the same  $\theta$  also satisfies

$$\alpha \int_0^{\infty} r\rho[C+x-\theta(C+x)]dF(x) - r\rho[\theta^{-1}(C)-C] - \gamma'(C) = 0 \quad (103.2)$$

that is, it satisfies

$$\alpha \int_0^{\infty} \rho[C+x-\theta(C+x)]dF(x) - \rho[\theta^{-1}(C)-C] - (1/r)\gamma'(C) = 0 \quad (103.3)$$

Hence, changing  $\rho(Y)$  to  $\rho^*(Y) = r\rho(Y)$  is equivalent in its effects on  $\hat{\theta}$  to changing  $\gamma'(C)$  to  $\gamma'^*(C) = (1/r)\gamma'(C)$ . Also, if  $\rho^*(Y) = r\rho(Y)$  and  $\gamma'^*(C) = r\gamma'(C)$ , then the same  $\theta$  is optimal under either  $\rho^*$ ,  $\gamma'^*$  or  $\rho$ ,  $\gamma'$ .

(2) If  $\theta$  satisfies

$$\alpha \int_0^{\infty} \rho^*[C+x-\theta(C+x)]dF(x) - \rho^*[\theta^{-1}(C)-C] - \gamma'(C) = 0 \quad (104.1)$$

where  $\rho^*(Y) = \rho(Y) + K$ , then the same  $\theta$  also satisfies

$$\alpha \int_0^{\infty} \rho[C+x-\theta(C+x)]dF(x) + \alpha K - \rho[\theta^{-1}(C)-C] - K - \gamma'(C) = 0 \quad (104.2)$$

that is, it satisfies

$$\alpha \int_0^{\infty} \rho[C+x-\theta(C+x)]dF(x) - \rho[\theta^{-1}(C)-C] - [\gamma'(C) + (1-\alpha)K] = 0 \quad (104.3)$$

Hence, changing  $\rho(Y)$  to  $\rho^*(Y) = \rho(Y) + K$  is equivalent (in its effects on  $\hat{\theta}$ ) to changing  $\gamma'(C)$  to

$$\gamma'^*(C) = \gamma'(C) + (1-\alpha)K \quad (104.4)$$

(3) If  $\theta$  satisfies

$$\alpha \int_0^{\infty} \rho^*[C+x-\theta(C+x)]dF(x) - \rho^*[\theta^{-1}(C)-C] - \gamma'(C) = 0 \quad (105.1)$$

where  $\rho^*(Y) = r\rho(Y) + K$ , then the same  $\theta$  also satisfies

$$\alpha \int_0^\infty r\rho[C+x-\theta(C+x)]dF(x) + \alpha K - r\rho[\theta^{-1}(C) - C] - K - \gamma'(C) = 0 \quad (105.2)$$

that is, it satisfies

$$\alpha \int_0^\infty \rho[C+x-\theta(C+x)]dF(x) - \rho[\theta^{-1}(C) - C] - (1/r)[\gamma'(C) + (1-\alpha)K] = 0 \quad (105.3)$$

Hence, changing  $\rho(Y)$  to  $\rho^*(Y) = r\rho(Y) + K$  is equivalent in its effects on  $\hat{\theta}$  to changing  $\gamma'(C)$  to

$$\gamma'^*(C) = (1/r)\gamma'(C) + (1/r)(1-\alpha)K \quad (105.4)$$

### NOTE 8.—RELATION BETWEEN OPTIMAL STORAGE RULES UNDER DIFFERENT YIELD DISTRIBUTIONS (SEE PAGE 50)

To simplify the notation here, consider  $S$ ,  $Y$ , and  $x$  to be measured as deviations from  $\mu$ . That is, wherever  $S$  appears in this note it means  $S - \mu$ , and similarly for  $Y$  and  $x$ . Hence, in this note,  $\mu = 0$ . Let  $G$  and  $F$  be alternative probability distributions of  $x$  such that if  $g$  and  $f$  are the respective probability density functions,  $g(rx) = (1/r)f(x)$ . Then  $G$  has the same mean  $\mu$  as  $F$ , and the standard deviation of  $G$  is  $r$  times the standard deviation of  $F$ , i. e.,  $\sigma_G = r\sigma_F$ .

If  $\hat{\theta}_G$  is the optimal storage rule under  $G$  it satisfies

$$\alpha \int_{-\infty}^\infty \rho[C+y-\hat{\theta}_G(C+y)]g(y)dy - \rho[\hat{\theta}_G^{-1}(C) - C] - \gamma'(C) = 0 \quad (26.5)$$

Now define  $\theta^*(S) = (1/r)\hat{\theta}_G(rS)$ . Then  $\theta^{*-1}(C) = (1/r)\hat{\theta}_G^{-1}(rC)$ , since if we set  $C = \theta^*(S) = (1/r)\hat{\theta}_G(rS)$  and solve for  $S$ , we have

$$rC = \hat{\theta}_G(rS) \quad (106)$$

$$\hat{\theta}_G^{-1}(rC) = rS \quad (106.1)$$

and

$$(1/r)\hat{\theta}_G^{-1}(rC) = S = \theta^{*-1}(C) \quad (106.2)$$

Also,

$$\hat{\theta}_G(S) = r\theta^*(S/r) \quad (106.3)$$

and

$$\hat{\theta}_G^{-1}(C) = r\theta^{*-1}(C/r) \quad (106.4)$$

Thus,  $\theta^*$  satisfies

$$\alpha \int_{-\infty}^\infty \rho[C+y-r\theta^*(C/r+y/r)]g(y)dy - \rho[r\theta^{*-1}(C/r) - C] - \gamma'(C) = 0 \quad (107)$$

Now let  $\rho^*(Y) = \rho(rY)$ , that is,  $\rho(Y) = \rho^*(Y/r)$ . Then  $\theta^*$  satisfies

$$\alpha \int_{-\infty}^{\infty} \rho^*[C/r + y/r - \theta^*(C/r + y/r)]g(y)dy - \rho^*[\theta^{*-1}(C/r) - C/r] - \gamma'(C) = 0 \quad (107.1)$$

But since this is true for any value of  $C$ , then if  $\gamma'(rC) = \gamma'(C)$  (for example, if  $\gamma'$  is constant),  $\theta^*$  satisfies

$$\alpha \int_{-\infty}^{\infty} \rho^*[C + y/r - \theta^*(C + y/r)]g(y)dy - \rho^*[\theta^{*-1}(C) - C] - \gamma'(C) = 0 \quad (107.2)$$

Now let  $y/r = x$ , so  $y = rx$  and  $g(y)dy = g(rx)rdx = (1/r)f(x)rdx = f(x)dx$ . Then  $\theta^*$  satisfies

$$\alpha \int_{-\infty}^{\infty} \rho^*[C + x - \theta^*(C + x)]f(x)dx - \rho^*[\theta^{*-1}(C) - C] - \gamma'(C) = 0 \quad (107.3)$$

Hence, to find  $\hat{\theta}_G$ , we find  $\theta^*$  which satisfies the last equation, and then  $\hat{\theta}_G(S) = r\theta^*(S/r)$  (where  $S$ , it is remembered, is here measured from  $\mu$ ).

#### NOTE 9.—PROOF OF THE METHOD OF OBTAINING THE OPTIMAL STORAGE RULE $\theta$ FOR THE CASE WHERE FUTURE OUTPUT IS CONSTANT (SEE PAGES 53-54)

For simplicity, the circumflex  $\wedge$  is omitted from  $\theta$  in this note, it being understood that we are dealing with an optimal rule. The segments of the rule are designated  $\theta_1, \theta_2, \theta_3, \dots$ , it being understood that these are segments of a single rule and not different rules for different years (as in the earlier notation). The initial point for segment  $\theta_1$  (the lower end of the segment) is  $(C_0, S_0)$ , where  $C_0 = 0$ . The terminal point for segment  $\theta_i$  (the upper end of the segment) is  $(C_i, S_i)$  ( $i = 1, 2, 3, \dots$ ). Our object is to determine the segments of the optimal rule,  $\theta_1, \theta_2, \dots$ , and, in particular, to determine the values of the segment connecting points  $(C_i, S_i)$  ( $i = 0, 1, 2, \dots$ ), given the discount factor  $\alpha$ , the (constant) marginal cost of storage  $\gamma'$ , and the marginal value function  $\rho(Y)$ .

We designate the constant future harvest by  $h$ , and define the linear operator  $D$  by  $DZ = \alpha Z - \gamma'$ . Starting with the "fundamental" equation for optimality of the storage rule  $\theta$ ,

$$\alpha \int_0^{\infty} \rho[C + x - \theta(C + x)]dF(x) - \gamma'(C) - \rho[\theta^{-1}(C) - C] = 0 \quad (26.1)$$

this becomes, for constant output  $h$  and constant marginal cost of storage  $\gamma'$ ,

$$\alpha \rho[C + h - \theta(C + h)] - \gamma' - \rho[\theta^{-1}(C) - C] = 0 \quad (26.6)$$

which may be rewritten as

$$\theta^{-1}(C) = C + \rho^{-1}\{\alpha\rho[C+h-\theta(C+h)]-\gamma'\} \quad (108)$$

This is the basic equation to be used in the derivation.

For  $C=0$ ,  $\theta^{-1}(C)=S_0$  (the intercept of  $\theta$  on the S-axis), so, from equation (108):

$$S_0 = \rho^{-1}\{\alpha\rho[h-\theta(h)]-\gamma'\} \quad (109)$$

If  $\rho' < 0$  and  $\alpha \leq 1$  and  $\gamma' \geq 0$ , then  $\theta(h)=0$ , since if  $\theta(h) > 0$ , then  $S_0 < h$ , which contradicts  $\theta(h) > 0$ . Therefore,

$$S_0 = \rho^{-1}[\alpha\rho(h)-\gamma'] = \rho^{-1}[D\rho(h)] \quad (41)$$

and we have determined the initial point  $(C_0, S_0)$  for the first segment  $\theta_1$ .

We thus have  $\theta(S)=0$  for  $S \leq S_0$ , which gives  $\theta(C+h)=0$  for  $C+h \leq S_0$  or  $C \leq S_0-h$ . It follows that [from equation (108)] for  $0 \leq C \leq S_0-h$ , the inverse storage rule  $\theta^{-1}(C)$  is given by

$$\theta_1^{-1}(C) = C + \rho^{-1}[\alpha\rho(C+h)-\gamma'] \quad (108.1)$$

where  $\theta_1$  is the first segment of the rule, and is completely defined by the expression given. The terminal point of this segment is:

$$C_1 = S_0 - h \quad (110.1)$$

$$S_1 = \theta_1^{-1}(S_0 - h) \quad (110.2)$$

$$= S_0 - h + \rho^{-1}[\alpha\rho(S_0) - \gamma'] \quad (110.3)$$

$$= S_0 - h + \rho^{-1}[D^2\rho(h)] \quad (110.4)$$

We thus have  $\theta(S)=\theta_1(S)$  for  $S_0 \leq S \leq S_1$ , which gives  $\theta(C+h)=\theta_1(C+h)$  for  $S_0 \leq C+h \leq S_1$ , or  $S_0-h \leq C \leq S_1-h$ . It follows (from equation (108)) that for  $S_0-h \leq C \leq S_1-h$ , the inverse storage rule  $\theta^{-1}(C)$  is given by

$$\theta_2^{-1}(C) = C + \rho^{-1}\{\alpha\rho[C+h-\theta_1(C+h)]-\gamma'\} \quad (108.2)$$

where  $\theta_2$  is the second segment of the rule, and is completely defined by the expression given. The terminal point of this segment is:

$$C_2 = S_1 - h \quad (111.1)$$

$$S_2 = \theta_2^{-1}(S_1 - h) \quad (111.2)$$

$$= S_1 - h + \rho^{-1}\{\alpha\rho[S_1-\theta_1(S_1)]-\gamma'\} \quad (111.3)$$

$$= S_1 - h + \rho^{-1}[\alpha\rho(S_1 - C_1) - \gamma'] \quad (111.4)$$

$$=S_1-h+\rho^{-1}\{\alpha\rho[\rho^{-1}D^2\rho(h)]-\gamma'\} \quad (111.5)$$

$$=S_1-h+\rho^{-1}[\alpha D^2\rho(h)-\gamma'] \quad (111.6)$$

$$=S_1-h+\rho^{-1}[D^3\rho(h)] \quad (111.7)$$

Continuing the proof, by induction: If  $\theta(S)=\theta_{i-1}(S)$  for  $S_{i-2}\leq S\leq S_{i-1}$ , where the terminal point of  $\theta_{i-1}$  is

$$C_{i-1}=S_{i-2}-h \quad (112.1)$$

$$S_{i-1}=S_{i-2}-h+\rho^{-1}[D^4\rho(h)] \quad (112.2)$$

then it follows [from equation (108)] that for  $S_{i-2}-h\leq C\leq S_{i-1}-h$ , the inverse storage rule  $\theta^{-1}(C)$  is given by

$$\theta_i^{-1}(C)=C+\rho^{-1}\{\alpha\rho[C+h\theta_{i-1}(C+h)]-\gamma'\} \quad (108.3)$$

where  $\theta_i$  is the  $i^{\text{th}}$  segment of the rule, and is completely defined by the expression given. The terminal point of the  $i^{\text{th}}$  segment is:

$$C_i=S_{i-1}-h \quad (44)$$

$$S_i=\theta_i^{-1}(S_{i-1}-h) \quad (113)$$

$$=S_{i-1}-h+\rho^{-1}\{\alpha\rho[S_{i-1}-\theta_{i-1}(S_{i-1})]-\gamma'\} \quad (113.1)$$

$$=S_{i-1}-h+\rho^{-1}[\alpha\rho(S_{i-1}-C_{i-1})-\gamma'] \quad (113.2)$$

$$=S_{i-1}-h+\rho^{-1}\{\alpha\rho[\rho^{-1}D^4\rho(h)]-\gamma'\} \quad (113.3)$$

$$=S_{i-1}-h+\rho^{-1}[D^{4+1}\rho(h)] \quad (42)$$

To complete the proof, we check that the segments are connected, that is, that the terminal point of the  $(i-1)^{\text{th}}$  segment lies on the  $i^{\text{th}}$  segment:

$$\theta_i^{-1}(C_{i-1})\stackrel{?}{=}C_{i-1}+\rho^{-1}\{\alpha\rho[C_{i-1}+h-\theta_{i-1}(C_{i-1}+h)]-\gamma'\} \quad (114)$$

that is,

$$S_{i-1}\stackrel{?}{=}C_{i-1}+\rho^{-1}\{\alpha\rho[S_{i-2}-\theta_{i-1}(S_{i-2})]-\gamma'\} \quad (114.1)$$

The expression on the right of the equality sign reduces to

$$C_{i-1}+S_{i-1}-S_{i-2}+h=S_{i-1} \quad (115)$$

This completes the proof.

It is clear from the expressions for  $\theta_i$  ( $i=1,2,\dots$ ) that, if  $\rho$  is linear, the storage rule segments  $\theta_i$  also are linear. It is fairly easy to write out explicitly the algebraic expressions for the consecutive segments. If the marginal value function  $\rho$  is not linear, the storage

rule segments are not linear, but can usually be adequately approximated by linear segments connecting the end points. If this approximation is felt not to be adequate, intermediate points along the segments can be computed using the expressions derived above.

**NOTE 10.—METHOD OF APPROXIMATING THE VALUE OF THE S-AXIS INTERCEPT K OF AN OPTIMAL STORAGE RULE (SEE PAGES 54-55)**

We wish to show that the S-axis intercept  $k$  of an optimal storage rule can be approximated by solving the following equation for  $k$ :  $k = K - \alpha a L(k)$ , where for simplicity we substitute the symbol  $a$  for the symbol  $\theta'_a$  defined on page 55, and the other symbols are defined on page 55.

With given  $a$  (by an a priori assumption about the average slope of the optimal storage rule), the optimal rule  $\theta$  can be approximated by the expression

$$\theta(S) = \begin{cases} a(S-k) & \text{for } S \geq k \\ 0 & \text{for } S \leq k \end{cases} \quad (116)$$

Then  $\theta^{-1}(C) = C/a + k$ .

If the marginal value function is linear, we use it directly, otherwise we approximate it by a linear function  $\rho(Y) = q - pY$ , where  $q$  and  $p$  are chosen to give, at  $Y = Ex$ , the same value of  $\rho$  and the same slope as that of the actual  $\rho$ .

With  $\rho(Y) = q - pY$  (actual or approximate), the basic equation for optimality of  $\theta$  becomes:

$$\theta^{-1}(C) = K + (1 + \alpha)C - \alpha \int_{k-C}^{\infty} \theta(C+x) dF(x) \quad (40)$$

where

$$K = \gamma'/p + (1 - \alpha)q/p + \alpha Ex \quad (40.1)$$

$$= Ex - (1 - \alpha)\rho(Ex)/\rho'(Ex) - \gamma'/\rho'(Ex) \quad (40.2)$$

The second expression for  $K$  is equivalent to the first, since  $p = -\rho'(Ex)$  and  $q = \rho(Ex) + pEx$ .

Using the approximation for  $\theta$  given by equation (116), equation (40) becomes  $C/a + k = K + (1 + \alpha)C - \alpha \int_{k-C}^{\infty} a(C+x-k) dF(x)$  (40.3)

so that, at  $C=0$ , we have

$$k = K - \alpha a \int_k^{\infty} (x-k) dF(x) = K - \alpha a L(k) \quad (13.1)$$

This completes the proof.

For the case where the actual  $\rho$  is not linear, a closer approximation to  $k$ , but one requiring more computational labor, can be obtained as follows:

We have

$$\theta^{-1}(C) = C + \rho^{-1} \left\{ \alpha \int_0^{\infty} \rho[C+x - \theta(C+x)] dF(x) - \gamma' \right\} \quad (117)$$

Using the approximation for  $\theta$  given by equation (116), this becomes

$$C/a + k = C + \rho^{-1} \left\{ \alpha \int_0^{k-C} \rho(C+x) dF(x) + \alpha \int_{k-C}^{\infty} \rho[(1-a)(C+x)] dF(x) - \gamma' \right\} \quad (117.1)$$

so that, at  $C=0$ , we have:

$$k = \rho^{-1} \left\{ \alpha \int_0^k \rho(x) dF(x) + \alpha \int_k^{\infty} \rho[(1-a)x + ak] dF(x) - \gamma' \right\} \quad (13.2)$$

The expression on the right side of the equality sign is a function of  $k$ , so that the equation can be solved for  $k$  by numerical methods. When  $\rho$  is linear, the above equation reduces to the simpler one,

$$k = K - \alpha a L(k) \quad (13.1)$$

### NOTE 11.—THE EQUILIBRIUM LEVEL (SEE PAGE 56)

If  $\theta(S)$  is continuous and  $0 \leq d\theta(S)/dS < r < 1$ , then consider the function

$$\Delta(C) = \int_0^{\infty} \theta(C+x) dF(x) - C \quad (118)$$

$$d\Delta(C)/dC = \int_0^{\infty} \theta'(C+x) dF(x) - 1 < r - 1 < 0 \quad (119)$$

Therefore, if  $\Delta(C^*) = 0$  for some value  $C^*$ , that value is unique. But if  $\theta(x_{\max}) > 0$ , then

$$\Delta(0) = \int_0^{\infty} \theta(x) dF(x) - 0 > 0 \quad (120)$$

and, since  $d\Delta(C)/dC < r - 1$ , therefore  $C^* > 0$  exists.

Also, for  $C_t < C^*$ ,  $\Delta(C_t) > 0$ , that is,  $\int_0^{\infty} \theta(C_t+x) dF(x) > C_t$ ; but  $\int_0^{\infty} \theta(C_t+x) dF(x) \leq C^*$ , since if this were not so, then we would have  $\int_0^{\infty} \theta(C_t+x) dF(x) > \int_0^{\infty} \theta(C^*+x) dF(x)$ , which violates the condition that  $\theta' \geq 0$ . Similarly, for  $C_t > C^*$ ,  $\Delta(C_t) < 0$ , that is,  $\int_0^{\infty} \theta(C_t+x) dF(x) < C_t$ ; but  $\int_0^{\infty} \theta(C_t+x) dF(x) \geq C^*$ .

Hence, we have the result that  $EC_{t+1}$  always lies between  $C_t$  and  $C^*$ .

**NOTE 12.—GENERALIZATION OF THE SOLUTION TO ALLOW FOR EXPORTS (OR IMPORTS) AND OTHER FACTORS**

The basic storage-rule solution can be modified in various ways to make it applicable to grains for which foreign trade is important. The modification chosen for a particular application depends on the circumstances of the particular case, on the amount of information available, and on any possible modification in the criterion of optimality which may be required.

The simplest situation is one in which a country is committed, as by an international agreement, to export (or import) a specified amount of the grain each year. In this case, the amount to be exported (or imported) is subtracted from (or added to) the total supply for the year and storage rules for the resulting domestic supply are obtained in exactly the same way as outlined for a purely domestic grain in the main text.

Another case is one in which foreign trade occurs in essentially free markets. Let  $Q_t$  be net exports in year  $t$ , where "net" exports means total exports minus total imports. Then the demand for net exports may be written, for example, as

$$Q_t = \phi_1(P_t, Z_t, u_Q) \quad (121)$$

where  $\phi_1$  is a function to be estimated empirically,  $P_t$  is the domestic price,  $Z_t$  is a vector of other demand-influencing variables, say  $Z_t = (Z_{t1}, \dots, Z_{tk})$ , and  $u_Q$  is a random variable.  $Z_t$  is written as a vector to simplify the notation. It would presumably include among its elements such variables as foreign incomes, defined and measured in some relevant way, foreign supplies of the grain, transportation costs, and so forth. If such variables can be suitably defined and measured, and the function  $\phi_1$  obtained, it may be incorporated into the storage-rule solution in a way outlined below. In situations where such empirical measurements are not feasible, the simplest approach is to treat net export demand in future years as fluctuating in a random way around a price-determined mean value, analogously to the way random fluctuations in domestic demand were introduced in pages 51-52.

That is, we write

$$Q_t = \phi_2(P_t, u_Q) \quad (122)$$

where  $\phi_2$  is a function to be estimated empirically and  $u_Q$  is a random variable whose probability distribution is estimated on the basis of past experience, analogously to the estimation of the probability distribution of future harvests. Similarly, we have a domestic demand function with a random component,

$$Y_t = \phi_3(P_t, u_Y) \quad (123)$$

where  $Y_t$  is domestic consumption.

Combining (adding) equations (122) and (123) gives the total demand function:

$$D_t = Y_t + Q_t = \phi_4(P_t, u_D) \quad (124)$$

If we accept the total public value as measured by the area under the total demand curve, the marginal value function  $\rho$  is obtained by solving equation (124) for  $P_t$ :

$$P_t = \rho(D_t, u_D) \quad (125)$$

The optimal storage rules are then obtained in the way described in pages 40-48, noting that in any year  $t$  the identity

$$D_t = Y_t + Q_t = S_t - C_t \quad (126)$$

must apply, that is,

$$P_t = \rho[(S_t - C_t), u_D] \quad (127)$$

Returning to the situation where equation (121) can be estimated, we may assume that a more precise domestic demand function than equation (123) is also estimatable, and write for domestic demand, say,

$$Y_t = \phi_t(P_t, Z_t, u_Y) \quad (128)$$

where the vector  $Z_t$  is expanded to include variables influencing domestic demand as well as those influencing foreign demand.<sup>32</sup> From equations (121) and (128), obtain the total demand function

$$D_t = Y_t + Q_t = \phi_t(P_t, Z_t, u_D) \quad (129)$$

and solve for  $P_t$  to get the marginal value function  $\rho$ :

$$P_t = \rho(D_t, Z_t, u) \quad (130)$$

or

$$P_t = \rho[(S_t - C_t), Z_t, u] \quad (131)$$

where the subscript  $D$  in  $u_D$  is dropped for simplicity.

Consider now the situation in any year  $t$ . The variable  $u$  may be treated as known for the current year, written  $U_t$ , and as a random variable with known distribution in each future year, say  $u_{t+j}$  ( $j > 0$ ). The problem now is the following:

<sup>32</sup> This notation is adopted for convenience. All it means is that some of the elements of  $Z$  will appear with zero coefficients in equation (121), and other elements will appear with zero coefficients in equation (128). We ignore here a possible difficulty arising from "endogeneity" in some of the elements of  $Z$ , such as might occur, for example, in a country a large part of whose national income depended on production or exports of the grain. One way around such a possible difficulty would be to restrict the choice of variables in  $Z$  to those which are largely exogenous and/or lagged or "predetermined". For example, rather than including *prices* of possible substitute commodities (which may be partly endogenous) in the demand equation, it would generally be better to use their *supplies*, which in any given year may, at least in many cases, be treated as largely predetermined. This also makes the resulting demand function a better approximation to the (inverse) marginal value function, as described on pages 13-15.

Given the storage rule  $\theta_{t+1}$  which is

- a) applicable in year  $t+1$ ;
- b) a function of  $S_{t+1}$ ,  $Z_{t+1}$ , and  $U_{t+1}$ ; and
- c) optimal by the accepted criterion;

to find the storage rule  $\theta_t$  which is

- a) applicable in year  $t$ ;
- b) a function of  $S_t$ ,  $Z_t$ , and  $U_t$ ; and
- c) optimal.

If this problem is solved, then optimal storage rules for any number of years  $n$  can be found by the backward-iterative procedure, starting with the  $n^{\text{th}}$  year, and working back till the required number of years is covered (for a finite time horizon) or until convergence is obtained (for the case of stationarity).

Again taking for total public value the area under the total demand curve, we obtain as the condition for optimality of  $\theta_t(S_t, Z_t, U_t)$ , given the optimality of  $\theta_{t+1}(S_{t+1}, Z_{t+1}, U_{t+1})$ , the following: for every value of  $S_t$ ,  $Z_t$  and  $U_t$ , the carryover  $C_t$  must satisfy:

$$\rho[(S_t - C_t), Z_t, U_t] = -\gamma'(C_t) + \alpha E \rho\{[C_t + x_{t+1} - \theta_{t+1}(C_t + x_{t+1}, Z_{t+1}, u_{t+1})], Z_{t+1}, u_{t+1}\} \quad (132)$$

where  $\gamma'(C_t)$  is the marginal cost of storage and the expectation operator  $E$  is taken over the distributions of  $x_{t+1}$  and  $u_{t+1}$ . (As in the earlier solutions, if the value of  $C_t$  which satisfies equation (132) is negative, the optimal carryover is zero.)

If equation (132) is solved for  $C_t$ , then  $C_t$  becomes a function of  $S_t$ ,  $Z_t$ ,  $U_t$ , and  $Z_{t+1}$ . The  $Z_{t+1}$  variables must be eliminated, since they are, in general, not observable in period  $t$ . We introduce "expectation functions" or "prediction equations" as follows:

$$Z_{t+1,1} = \epsilon_1(Z_{t1}, \dots, Z_{tk}, v_1) \quad (133.1)$$

$$\vdots$$

$$Z_{t+1,k} = \epsilon_k(Z_{t1}, \dots, Z_{tk}, v_k) \quad (133.k)$$

where the functions  $\epsilon_1, \dots, \epsilon_k$  and the distributions of the random variables  $v_1, \dots, v_k$  are to be empirically estimated.<sup>23</sup>

Equations (133.1)–(133.k) may be summarized in vector notation as

$$Z_{t+1} = \epsilon(Z_t, v) \quad (133)$$

<sup>23</sup> The  $Z$  vector is possibly again expanded to include some prediction variables in addition to those already included as demand-determining variables. Again this is simply a matter of notational convenience. Those elements of  $Z$  which are irrelevant in any particular equation are considered to have zero coefficients therein. It may be that in one or more of equations (133.1)–(133.k), all of  $Z_{t1}, \dots, Z_{tk}$  appear with zero coefficients. If this should happen for, say, equation (133.j), it simply means that  $Z_{t+1,j}$  must, on the basis of available empirical data, be treated as a random variable whose distribution is that of  $v_j$ .

For somewhat greater generality, we may also introduce a prediction equation for output in period  $t+1$ :

$$X_{t+1} = \epsilon_x(Z_t, w) \quad (134)$$

where the function  $\epsilon_x$  and the distribution of the random variable  $w$  are estimated empirically, and  $Z_t$  now includes elements, for example lagged prices or acreage controls, which may aid in predicting  $X_{t+1}$ .<sup>34</sup>

Substituting (133) and (134) into (132) gives

$$\rho[(S_t - C_t), Z_t, U_t] = -\gamma'(C_t) + \alpha E\rho\{[C_t + \epsilon_x(Z_t, w) - \theta_{t+1}(C_t + \epsilon_x(Z_t, w), \epsilon(Z_t, v), u_{t+1})], \epsilon(Z_t, v), u_{t+1}\} \quad (135)$$

where the expectation operator  $E$  is taken as the integral over the distributions of  $u_{t+1}$ ,  $w$ , and  $v = (v_1, \dots, v_k)$ . Solving equation (135) for  $C_t$  gives  $C_t$  as a function of  $S_t$ ,  $Z_t$ , and  $U_t$ : the desired optimal storage rule for period  $t$ :

$$C_t = \theta_t(S_t, Z_t, U_t) \quad (136)$$

So far we have considered only cases where

a) the criterion of optimality is determined by taking total public value as equal to the area under the total demand curve, and

b) exports are price-determined in a free market. The methods can also be modified to allow for other kinds of criteria and/or possibly other institutional arrangements. In general, we can write total public value as a function, in each period  $t$ , of quantity consumed domestically and (net) quantity exported, say

$$\delta_t = \delta_t(Y_t, Q_t) \quad (137)$$

where we omit, for simplicity, the possibility of random components and/or other determining variables; these can be reintroduced in a way analogous to the procedures outlined in the preceding paragraphs. For example, total public value might be defined as the area under the domestic demand curve (a function of  $Y_t$ ) plus total revenue from exports (a function of  $Q_t$ ). Taking into account the identity (126), equation (137) becomes

$$\delta_t = \delta_t[(S_t - C_t - Q_t), Q_t] \quad (137.1)$$

<sup>34</sup> In summary, then, the vector  $Z_t$  consists of variables which are observable in period  $t$  and which:

- affect domestic demand in period  $t$ ;
- affect net export demand in period  $t$ ;
- affect output in period  $t+1$ ;
- are useful for predicting elements of  $Z_{t+1}$ ; and
- are preferably largely exogenous or predetermined.

It is clear that most of the elements of  $Z_t$  will have zero coefficients in most of the equations in which  $Z_t$  appears.

That is, for given  $S_t$ ,  $\delta_t$  is a function of  $C_t$  and  $Q_t$ . There are now two principal possibilities open, depending on the institutional setting:

a) If exports are price-determined in a free market, then an additional relation between  $Y_t$  and  $Q_t$  is established; that is, equations (122) and (123) can be combined (eliminating the price variable) to give, say

$$Q_t = \phi_7(Y_t) \quad (138)$$

(omitting the random components for simplicity). Combining equations (137), (126) and (138) gives total public value as a function of supply and carryout,

$$\delta_t = \phi_8(S_t, C_t) \quad (137.2)$$

which can then be used directly in the method of pages 40-44, or, if  $\delta$  is continuous and differentiable, the method of pages 44-48.

b) Alternatively, equation (137.1) may be looked on as a function with two variables which are "controllable" by a "policy maker," namely  $C_t$  and  $Q_t$ . This would in general imply a "two price" system, with the necessity of adding an additional variable for the export price, say  $P_t^e$ :

$$\delta_t = \delta_t[(S_t - C_t - Q_t), Q_t, P_t^e] \quad (137.3)$$

$P_t^e$  may be related to  $Q_t$  by a function analogous to equations (122) or (121), or, if the country's exports are small relative to total world supply,  $P_t^e$  may be treated as a random or partly predictable variable independent of  $Q_t$ . Then the solution proceeds by a generalization of the method of pages 40-44; at each step the expectation operator is taken over the distributions of both future output and future export price; and the maximization is with respect to both  $C$  and  $Q$ , thus leading to a set of "storage rules" and "export rules," each of which is a function of current supply and current export price. However, in this case the resulting solutions may not always be unique.